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FULL LENGTH PAPER

SDPNAL+: a majorized semismooth Newton-CG augmented Lagrangian method for semidefinite programming with nonnegative constraints

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Abstract In this paper, we present a majorized semismooth Newton-CG augmented Lagrangian method, called SDPNAL+, for semidefinite programming (SDP) with partial or full nonnegative constraints on the matrix variable. SDPNAL+ is a much enhanced version of SDPNAL introduced by Zhao et al. (SIAM J Optim 20:1737–1765, 2010) for solving generic SDPs. SDPNAL works very efficiently for nondegenerate SDPs but may encounter numerical difficulty for degenerate ones. Here we tackle this numerical difficulty by employing a majorized semismooth Newton-CG augmented Lagrangian method coupled with a convergent 3-block alternating direction method of multipliers introduced recently by Sun et al. (SIAM J Optim, to appear). Numerical results for various large scale SDPs with or without nonnegative constraints show that the proposed method is not only fast but also robust in obtaining accurate solutions. It outperforms, by a significant margin, two other competitive publicly available first order methods based codes: (1) an alternating direction method of

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multipliers based solver called SDPAD by Wen et al. (Math Program Comput 2:203–230, 2010) and (2) a two-easy-block-decomposition hybrid proximal extragradient method called 2EBD-HPE by Monteiro et al. (Math Program Comput 1–48, 2014). In contrast to these two codes, we are able to solve all the 95 difficult SDP problems arising from the relaxations of quadratic assignment problems tested in SDPNAL to an accuracy of 10^{-6} efficiently, while SDPAD and 2EBD-HPE successfully solve 30 and 16 problems, respectively. In addition, SDPNAL+ appears to be the only viable method currently available to solve large scale SDPs arising from rank-1 tensor approximation problems constructed by Nie and Wang (SIAM J Matrix Anal Appl 35:1155–1179, 2014). The largest rank-1 tensor approximation problem we solved (in about 14.5 h) is `nonsym(21,4)`, in which its resulting SDP problem has matrix dimension $n = 9261$ and the number of equality constraints $m = 12{,}326{,}390$.

Keywords Semidefinite programming · Degeneracy · Augmented Lagrangian · Semismooth Newton-CG method

Mathematics Subject Classification 90C06 · 90C22 · 90C25 · 65F10

1 Introduction

Let $\mathcal{K}$ be a pointed closed convex cone whose interior $int(\mathcal{K}) \neq \emptyset$ and $\mathcal{P}$ be a polyhedral convex cone in a finite-dimensional Euclidean space $\mathcal{X}$ such that $\mathcal{K} \cap \mathcal{P}$ is non-empty. For any cone $\mathcal{C} \subseteq \mathcal{X}$, we denote the dual cone of $\mathcal{C}$ by $\mathcal{C}^*$. For any closed convex set $\mathcal{C} \subseteq \mathcal{X}$, we denote the metric projection of $\mathcal{X}$ onto $\mathcal{C}$ by $\Pi_{\mathcal{C}}(\cdot)$ and the tangent cone of $\mathcal{C}$ at $X \in \mathcal{C}$ by $\mathcal{T}_{\mathcal{C}}(X)$, respectively. We will make extensive use of the Moreau decomposition theorem in [11], which states that $X = \Pi_{\mathcal{C}}(X) - \Pi_{\mathcal{C}^*}(-X)$ for any $X \in \mathcal{X}$ and any closed convex cone $\mathcal{C} \subseteq \mathcal{X}$. Let $\mathcal{S}^n$ be the space of $n \times n$ real symmetric matrices and $\mathcal{S}^n_+$ be the cone of positive semidefinite matrices in $\mathcal{S}^n$. In this paper, we focus on the case where $\mathcal{X} = \mathcal{S}^n$, $\mathcal{K} = \mathcal{K}^* = \mathcal{S}^n_+$. We are particularly interested in the case where $\mathcal{P} = \mathcal{S}^n_{\geq 0}$, the cone of $n \times n$ real symmetric matrices whose elements are all nonnegative, though the algorithm which we will design later is also applicable to other cases. For any matrix $X \in \mathcal{S}^n$, we use $X \succ 0$ to indicate that X is a real symmetric positive definite matrix.

Consider the semidefinite programming (SDP) with an additional polyhedral cone constraint, which we name as SDP+:

$$\text{(P)} \quad \max\{\langle -C,\ X\rangle \mid \mathcal{A}(X) = b, \quad X \in \mathcal{K}, \quad X \in \mathcal{P}\}, \tag{1}$$

where $b \in \Re^m$ and $C \in \mathcal{X}$ are given data, $\mathcal{A} : \mathcal{X} \to \Re^m$ is a given linear map whose adjoint is denoted as $\mathcal{A}^*$. Note that $\mathcal{P} = \mathcal{X}$ is allowed in (1), in which case there is no additional polyhedral cone constraint imposed on X. We assume that the matrix $\mathcal{A}\mathcal{A}^*$ is invertible, i.e., $\mathcal{A}$ is surjective. The dual of (P) is given by

$$\text{(D)} \quad \min\{\langle -b,\ y\rangle \mid \mathcal{A}^*(y) + S + Z = C, \quad S \in \mathcal{K}^*, \quad Z \in \mathcal{P}^*\}. \tag{2}$$

The optimality conditions (KKT conditions) for (P) and (D) can be written as follows:

$$\begin{cases} \mathcal{A}(X) - b = 0, \; \mathcal{A}^*(y) + S + Z - C = 0, \\ \langle X, \; S \rangle = 0, \quad X \in \mathcal{K}, \quad S \in \mathcal{K}^*, \quad \langle X, \; Z \rangle = 0, \quad X \in \mathcal{P}, \quad Z \in \mathcal{P}^*. \end{cases} \tag{3}$$

In order for the KKT conditions (3) to have solutions, throughout this paper we make the following blanket assumption.

Assumption 1 (a) For problem (P), there exists a feasible solution $X_0 \in \mathcal{S}^n_+$ such that

$$\mathcal{A}(X_0) = b, \quad X_0 \succ \mathbf{0}, \quad X_0 \in \mathcal{P}. \tag{4}$$

(b) For problem (D), there exists a feasible solution $(y_0, S_0, Z_0) \in \Re^m \times \mathcal{S}^n_+ \times \mathcal{S}^n$ such that

$$\mathcal{A}^*(y_0) + S_0 + Z_0 = C, \quad S_0 \succ 0, \quad Z_0 \in \mathcal{P}^*. \tag{5}$$

It is known from convex analysis (e.g, [3, Corollary 5.3.6]) that under Assumption 1, the strong duality for (P) and (D) holds and the KKT conditions (3) have solutions.

For a given $\sigma > 0$, define the augmented Lagrangian function for the dual problem (D) as follows:

$$\begin{aligned} L_\sigma(y, S, Z; X) &= \langle -b, \; y \rangle + \langle X, \; \mathcal{A}^* y + S + Z - C \rangle + \frac{\sigma}{2} \|\mathcal{A}^* y + S + Z - C\|^2 \\ &= \langle -b, \; y \rangle + \frac{\sigma}{2} \|\mathcal{A}^* y + S + Z + \sigma^{-1} X - C\|^2 - \frac{1}{2\sigma} \|X\|^2, \end{aligned} \tag{6}$$

where $X \in \mathcal{X}$, $y \in \Re^m$, $S \in \mathcal{K}^*$, $Z \in \mathcal{P}^*$. We can consider the following inexact augmented Lagrangian method to solve (D). Specifically, given $\sigma_0 > 0$, $(y^0, S^0, Z^0) \in \Re^m \times \mathcal{K}^* \times \mathcal{P}^*$, perform the following steps at the $(k+1)$th iteration:

$$\begin{cases} (y^{k+1}, S^{k+1}, Z^{k+1}) \approx \arg\min\{L_{\sigma_k}(y, S, Z; X^k) \mid y \in \Re^m, \; S \in \mathcal{K}^*, \; Z \in \mathcal{P}^*\}, & \text{(7a)} \\ X^{k+1} = X^k + \sigma_k(\mathcal{A}^* y^{k+1} + S^{k+1} + Z^{k+1} - C), & \text{(7b)} \end{cases}$$

where $\sigma_k \in (0, +\infty)$, $k = 0, 1, \ldots$ For a general discussion on the convergence of the augmented Lagrangian method for solving convex optimization problems and beyond, see [18,19].

Note that problem (P) can be reformulated as a standard SDP in the primal form by replacing the constraint $X \in \mathcal{P}$ with two constraints $X - Y = 0$ and $Y \in \mathcal{P}$. In [26], SDPNAL introduced by Zhaoet al. is applied to solve such a reformulated problem. It works quite well for nondegenerate SDPs, especially those without the constraint $X \in \mathcal{P}$. However, many of the tested SDPs (with the constraint $X \in \mathcal{P}$) in [26] are degenerate and SDPNAL is unable to solve those problems efficiently. Motivated by our desire to overcome the aforementioned difficulty in solving degenerate SDPs and to improve the performance of SDPNAL, we present here a majorized semismooth Newton-CG augmented Lagrangian method by directly working on (P) instead of its reformulated problem. We call this new method SDPNAL+ since it is a much enhanced version of SDPNAL and it is designed for SDP+ problems (P). We should emphasize that the technique of majorization plays a pivotal role in solving the inner problem (7a), which leads to an alternating minimization between the blocks (y, S) and Z.

The remaining parts of this paper are organized as follows. In Sect. 2, we introduce a majorized semismooth Newton-CG method for solving the inner minimization problems of the augmented Lagrangian method and analyze the convergence for solving these inner problems. Section 3 presents the SDPNAL+ dual approach. Section 4 is on numerical issues. There we report numerical results for a variety of SDP+ and SDP problems. We make an extensive numerical comparison with two other competitive first order methods based codes: (1) an alternating direction method of multiplier (ADMM) based solver called SDPAD by Wen et al. [25] and (2) a two-easy-block-decomposition hybrid proximal extragradient method called 2EBD-HPE by Monteiro et al. [10]. Numerical results show that SDPNAL+ is both fast and robust in achieving accurate solutions.

For the first time, we are able to solve all the 95 difficult SDP problems arising from the relaxations of quadratic assignment problems (QAPs) tested in SDPNAL to an accuracy of 10^{-6} efficiently, while SDPAD and 2EBD-HPE successfully solve 30 and 16 problems, respectively. In addition, SDPNAL+ appears to be the only viable method currently available to solve large scale SDPs arising from rank-1 tensor approximation problems constructed by Nie and Wang [12]. The largest rank-1 tensor approximation problem solved is `nonsym(21,4)`, in which its resulting SDP problem has matrix dimension $n = 9261$ and the number of equality constraints $m = 12{,}326{,}390$. Finally, in order to demonstrate the power of the proposed majorized semismooth Newton-CG procedure, we list the numerical results by only running the convergent ADMM with 3-block constraints (ADMM+ in short) introduced by Sun et al. [22]. As one may observe, although ADMM+ outperforms both SDPAD and 2EBD-HPE, it can still encounter numerical difficulty in solving some hard problems such as those arising from QAPs to high accuracy. The superior numerical performance of SDPNAL+ over solvers based purely on first order methods such as SDPAD and 2EBD-HPE clearly shows the necessity of exploiting second order methods such as the semismooth Newton-CG method in order to solve hard SDP+ and SDP problems to high accuracy efficiently. While there has been a recent focus on using first order methods such as those based on ADMM or accelerated proximal gradient methods to solve structured convex optimization problems arising from machine learning and statistics, the extensive numerical results we obtained here for matrix conic programming problems serve to demonstrate that second order methods with good local convergence property are essential, if used wisely, for mitigating the inherent slow local convergence of first order methods, especially on difficult problems.

2 A majorized semismooth Newton-CG method for inner problems

Let $\sigma > 0$ and $\widetilde{X} \in \mathcal{S}^n$ be fixed. In this section we will present a majorized semismooth Newton-CG method for solving the following inner problems involved in the augmented Lagrangian method (7a):

$$\min\{\phi(y, S, Z) := L_\sigma(y, S, Z; \widetilde{X}) \mid y \in \Re^m, \quad S \in \mathcal{K}^*, \quad Z \in \mathcal{P}^*\}. \tag{8}$$

Note that problem (8) is the dual of the following problem:

$$\max\left\{\langle -C, X\rangle - \frac{1}{2\sigma}\|X - \widetilde{X}\|^2 \mid \mathcal{A}(X) = b, \quad X \in \mathcal{K}, \quad X \in \mathcal{P}\right\}. \tag{9}$$

Since the objective function in (9) is strongly concave, (9) has a unique optimal solution. In order for its dual problem (8) to have a bounded solution set, we need the following generalized Slater condition.

Assumption 2 There exists a positive definite matrix $X_0 \in \mathcal{S}^n_+ \cap relint(\mathcal{P})$ such that

$$\mathcal{A}(\mathcal{T}_{\mathcal{P}}(X_0)) = \Re^m, \quad X_0 \succ 0, \tag{10}$$

where $relint(\mathcal{P})$ denotes the relative interior of $\mathcal{P}$.

Note that in (10), $\mathcal{T}_{\mathcal{P}}(X_0)$ is actually a linear subspace of $\mathcal{S}^n$ as X_0 is assumed to be in the relative interior part of the polyhedral cone $\mathcal{P}$. When $\mathcal{P} = \mathcal{S}^n$, Assumption 2 is equivalent to saying that

$$\begin{cases} \mathcal{A} : \mathcal{S}^n \to \Re^m \text{ is onto,} \\ \exists\, X_0 \in \mathcal{S}^n_+ \text{ such that } \mathcal{A}(X_0) = b, \quad X_0 \succ \mathbf{0}. \end{cases} \tag{11}$$

From [17, Theorems 17 and 18], we have the following useful lemma.

Lemma 2.1 *Suppose that Assumption 2 holds. Then for any* $\alpha \in \Re$, *the level set* $\mathcal{L}_\alpha := \{(y, S, Z) \in \Re^m \times \mathcal{K}^* \times \mathcal{P}^* \mid \phi(y, S, Z) \leq \alpha\}$ *is a closed and bounded convex set.*

2.1 A majorized semismooth Newton-CG method

Consider $(\tilde{y}, \widetilde{S}, \widetilde{Z}) \in \arg\min\{\phi(y, S, Z) \mid y \in \Re^m, S \in \mathcal{K}^*, Z \in \mathcal{P}^*\}$. Let

$$\widehat{C} = C - \sigma^{-1}\widetilde{X}.$$

Then we must have $\widetilde{Z} = \Pi_{\mathcal{P}^*}(\widehat{C} - \mathcal{A}^*\tilde{y} - \widetilde{S})$. Therefore, problem (8) is equivalent to the following optimization problem:

$$\min\left\{\Phi(y, S) := \langle -b, \ y\rangle + \frac{\sigma}{2}\|\Pi_{\mathcal{P}}(\mathcal{A}^*y + S - \widehat{C})\|^2 \mid y \in \Re^m, \quad S \in \mathcal{K}^*\right\}. \tag{12}$$

In order to introduce our majorized semismooth Newton-CG method for solving (12), we need to majorize the second part of the objective function in (12) by a convex, but not necessarily strongly convex, quadratic function. Specifically, for given $(y^l, S^l) \in \Re^m \times \mathcal{K}^*$ and $l \geq 0$, since

$$\begin{aligned}\|\Pi_{\mathcal{P}}(\mathcal{A}^* y + S - \widehat{C})\|^2 &\leq \|\Pi_{\mathcal{P}}(\mathcal{A}^* y^l + S^l - \widehat{C})\|^2 + \|\mathcal{A}^* y + S - \mathcal{A}^* y^l - S^l\|^2 \\ &\quad + 2\langle \Pi_{\mathcal{P}}(\mathcal{A}^* y^l + S^l - \widehat{C}),\ \mathcal{A}^* y + S - \mathcal{A}^* y^l - S^l\rangle \\ &= \|\mathcal{A}^* y + S + Z^l - \widehat{C}\|^2,\end{aligned}$$

where $Z^l := \Pi_{\mathcal{P}^*}(\widehat{C} - \mathcal{A}^* y^l - S^l)$, we know that for $(y, S) \in \Re^m \times \mathcal{S}^n$,

$$\begin{aligned}\Phi(y, S) &\leq \langle -b,\ y\rangle + \frac{\sigma}{2}\|\mathcal{A}^* y + S + Z^l - \widehat{C}\|^2 \\ &= \Psi_l(y, S) := \langle -b,\ y\rangle + \frac{\sigma}{2}\|\mathcal{A}^* y + S + \sigma^{-1}\widetilde{X} - \widetilde{C}^l\|^2, \end{aligned} \tag{13}$$

where $\widetilde{C}^l := C - Z^l$. Thus Ψ_l is a majorization function of Φ at (y^l, S^l) because $\Psi_l(y^l, S^l) = \Phi(y^l, S^l)$ and $\Psi_l(y, S) \geq \Phi(y, S) \forall (y, S) \in \Re^m \times \mathcal{S}^n$. In order to find an optimal solution for problem (12), for $l = 0, 1, \ldots$, we solve the following problem

$$\min\{\Psi_l(y, S) \mid y \in \Re^m, \quad S \in \mathcal{K}^*\}. \tag{14}$$

Observe that if $(\tilde{y}, \widetilde{S}) \in \arg\min\{\Psi_l(y, S) \mid y \in \Re^m,\ S \in \mathcal{K}^*\}$, then we must have $\widetilde{S} = \Pi_{\mathcal{K}^*}(\widetilde{C}^l - \mathcal{A}^* \tilde{y} - \sigma^{-1}\widetilde{X})$. Thus we can compute y^{l+1} and S^{l+1} simultaneously as follows:

$$\begin{cases} y^{l+1} \in \arg\min\left\{\varphi_l(y) := \langle -b,\ y\rangle + \frac{\sigma}{2}\|\Pi_{\mathcal{K}}(\mathcal{A}^* y + \sigma^{-1}\widetilde{X} - \widetilde{C}^l)\|^2 \mid y \in \Re^m\right\}, & \text{(15a)} \\ S^{l+1} = \Pi_{\mathcal{K}^*}(\widetilde{C}^l - \mathcal{A}^* y^{l+1} - \sigma^{-1}\widetilde{X}). & \text{(15b)} \end{cases}$$

Note that we can only solve problem (15a) inexactly by an iterative method. Here we will introduce a semismooth Newton-CG (SNCG) method for solving (15a). Specifically, for fixed $\widetilde{X}, \widetilde{C} \in \mathcal{S}^n$, we need to consider the following problem of the form

$$\min\left\{\varphi(y) := \langle -b,\ y\rangle + \frac{\sigma}{2}\|\Pi_{\mathcal{K}}(\mathcal{A}^* y + \sigma^{-1}\widetilde{X} - \widetilde{C})\|^2 \mid y \in \Re^m\right\}. \tag{16}$$

The objective function in (16) is continuously differentiable and solving (16) is equivalent to solving the following nonsmooth equation:

$$\nabla\varphi(y) = \mathcal{A}\Pi_{\mathcal{K}}(\widetilde{X} + \sigma(\mathcal{A}^* y - \widetilde{C})) - b = 0, \quad y \in \Re^m. \tag{17}$$

Since $\Pi_{\mathcal{K}}(\cdot)$ is strongly semismooth [21], we can design a SNCG method as in [26] to solve (17), and expect fast superlinear or even quadratic convergence.

Let $\tilde{y} \in \Re^m$ be fixed. Consider the following eigenvalue decomposition:

$$\widetilde{X} + \sigma(\mathcal{A}^*\tilde{y} - \widetilde{C}) = Q\,\Gamma_{\tilde{y}}\,Q^{\mathrm{T}}, \tag{18}$$

where $Q \in \mathcal{R}^{n\times n}$ is an orthogonal matrix whose columns are eigenvectors, and $\Gamma_{\tilde{y}}$ is the diagonal matrix of eigenvalues with the diagonal elements arranged in the nonincreasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Define the following index sets

$$\alpha := \{i \mid \lambda_i > 0\}, \quad \bar{\alpha} := \{i \mid \lambda_i \leq 0\}.$$

We define the operator $W^0_{\tilde{y}} : \mathcal{S}^n \to \mathcal{S}^n$ by

$$W^0_{\tilde{y}}(H) := Q(\Sigma \circ (Q^{\mathrm{T}} H Q))Q^{\mathrm{T}}, \quad H \in \mathcal{S}^n, \tag{19}$$

where "$\circ$" denotes the Hadamard product of two matrices and

$$\Sigma = \begin{bmatrix} E_{\alpha\alpha} & \nu_{\alpha\bar{\alpha}} \\ \nu^{\mathrm{T}}_{\alpha\bar{\alpha}} & 0 \end{bmatrix}, \quad \nu_{ij} := \frac{\lambda_i}{\lambda_i - \lambda_j}, \quad i \in \alpha, \quad j \in \bar{\alpha}, \tag{20}$$

where $E_{\alpha\alpha} \in \mathcal{S}^{|\alpha|}$ is the matrix of ones. Define $V^0_{\tilde{y}} : \Re^m \to \mathcal{S}^n$ by

$$V^0_{\tilde{y}} d := \sigma\mathcal{A}[\,Q(\Sigma \circ (\,Q^{\mathrm{T}}(\mathcal{A}^* d)\,Q))\,Q^{\mathrm{T}}], \quad d \in \Re^m. \tag{21}$$

For any $y \in \Re^m$, define

$$\hat{\partial}^2\varphi(y) := \sigma\mathcal{A}\,\partial\Pi_{\mathcal{K}}(\widetilde{X} + \sigma(\mathcal{A}^* y - \widetilde{C}))\mathcal{A}^*, \tag{22}$$

where $\partial\Pi_{\mathcal{K}}(\widetilde{X}+\sigma(\mathcal{A}^* y-\widetilde{C}))$ is the Clarke subdifferential of $\Pi_{\mathcal{K}}(\cdot)$ at $\widetilde{X}+\sigma(\mathcal{A}^* y-\widetilde{C})$. Note that from [9], we know that

$$\hat{\partial}^2\varphi(\tilde{y})h = \partial^2\varphi(\tilde{y})h \quad \forall\, h \in \Re^m, \tag{23}$$

where $\partial^2\varphi(\tilde{y})$ denotes the generalized Hessian of φ at $\tilde{y}$, i.e., the Clarke subdifferential of $\nabla\varphi$ at $\tilde{y}$. However, note that (23) does not mean that $\hat{\partial}^2\varphi(\tilde{y}) = \partial^2\varphi(\tilde{y})$. Actually, it is unclear to us whether the latter holds. Fortunately, from Pang et al. [14, Lemma 11] we know that

$$W^0_{\tilde{y}} \in \partial\Pi_{\mathcal{K}}(\widetilde{X} + \sigma(\mathcal{A}^*\tilde{y} - \widetilde{C}))$$

and thus $V^0_{\tilde{y}} = \sigma\mathcal{A}W^0_{\tilde{y}}\mathcal{A}^* \in \hat{\partial}^2\varphi(\tilde{y})$.

Now we will introduce the SNCG algorithm for solving (16). Choose $y^0 \in \Re^m$. Then the algorithm can be stated as follows.

Algorithm SNCG: A Semismooth Newton-CG Algorithm (SNCG($y^0, \widetilde{X}, \sigma$)).

Given $\mu \in (0, 1/2)$, $\bar{\eta} \in (0, 1)$, $\tau \in (0, 1]$, $\tau_1, \tau_2 \in (0, 1)$, and $\delta \in (0, 1)$. Perform the jth iteration as follows.

Step 1. Given a maximum number of CG iterations $N_j > 0$, compute

$$\eta_j := \min(\bar{\eta}, \|\nabla\varphi(y^j)\|^{1+\tau}).$$

Apply the conjugate gradient (CG) algorithm ($CG(\eta_j, N_j)$), to find an approximation solution d^j to

$$(V_j + \varepsilon_j I)\, d = -\nabla\varphi(y^j), \tag{24}$$

where $V_j \in \hat{\partial}^2\varphi(y^j)$ is defined as in (21) and $\varepsilon_j := \tau_1 \min\{\tau_2, \|\nabla\varphi(y^j)\|\}$.

Step 2. Set $\alpha_j = \delta^{m_j}$, where m_j is the first nonnegative integer m for which

$$\varphi(y^j + \delta^m d^j) \le \varphi(y^j) + \mu\delta^m \langle \nabla\varphi(y^j), d^j \rangle. \tag{25}$$

Step 3. Set $y^{j+1} = y^j + \alpha_j\, d^j$.

The convergence results for the above SNCG algorithm are stated in Theorems 2.2 and 2.3 below. We shall omit the proofs as they can be proved in the same fashion as in [26, Theorems 3.4 and 3.5].

Theorem 2.2 *Suppose that Assumption 2 holds. Then Algorithm SNCG generates a bounded sequence $\{y^j\}$ and any accumulation point $\hat{y}$ of $\{y^j\}$ is an optimal solution to problem* (16).

Theorem 2.3 *Suppose that Assumption 2 holds. Let $\hat{y}$ be an accumulation point of the infinite sequence $\{y^j\}$ generated by Algorithm SNCG for solving the problem* (16)*. Suppose that at each step $j \ge 0$, when the CG algorithm terminates, the tolerance η_j is achieved* (*e.g., when $N_j = m + 1$*)*, i.e.,*

$$\|\nabla\varphi(y^j) + (V_j + \varepsilon_j I)\, d^j\| \le \eta_j. \tag{26}$$

Assume that the constraint nondegenerate condition

$$\mathcal{A}\,\mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\widehat{W})) = \Re^m \tag{27}$$

holds at $\widehat{W} := \Pi_{\mathcal{K}}(\widetilde{X} + \sigma(\mathcal{A}^*\hat{y} - \widetilde{C}))$, where $\mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\widehat{W}))$ denotes the lineality space of $\mathcal{T}_{\mathcal{K}}(\widehat{W})$. Then the whole sequence $\{y^j\}$ converges to $\hat{y}$ and

$$\|y^{j+1} - \hat{y}\| = O(\|y^j - \hat{y}\|^{1+\tau}). \tag{28}$$

For notational convenience, for any $y \in \Re^m$ and $S \in \mathcal{S}^n$, let $Y := (y, S)$. Define the linear map $\mathcal{M} : \Re^m \times \mathcal{S}^n \to \mathcal{S}^n$ by

$$\mathcal{M}Y := \mathcal{A}^* y + S \quad \forall\, Y = (y, S) \in \Re^m \times \mathcal{S}^n. \tag{29}$$

Let $B := (b, 0) \in \Re^m \times \mathcal{S}^n$ and $\mathcal{C} := \Re^m \times \mathcal{K}^*$. Then problem (12) is equivalent to

$$\min \left\{ \Phi(Y) := \langle -B,\ Y \rangle + \frac{\sigma}{2} \| \Pi_{\mathcal{P}}(\mathcal{M}Y + \sigma^{-1}\widetilde{X} - C) \|^2 \mid Y \in \mathcal{C} \right\} \tag{30}$$

and the function $\Psi_l(y, S)$ in (13) can be rewritten as

$$\Psi_l(Y) = \Psi_l(y, S) = -\langle B,\ Y \rangle + \frac{\sigma}{2} \| \mathcal{M}Y + \sigma^{-1}\widetilde{X} - C + Z^l \|^2.$$

Furthermore, $\widehat{Y} = (\hat{y}, \widehat{S})$ is an optimal solution of

$$\min\{\Psi_l(Y) \mid Y \in \mathcal{C}\} \tag{31}$$

if and only if $\hat{y}$ is an optimal solution of problem (15a) and $\widehat{S} = \Pi_{\mathcal{K}^*}(\widetilde{C}^l - \mathcal{A}^*\hat{y} - \sigma^{-1}\widetilde{X})$. Given $\xi_1 \in (0, 1)$ and $\xi_2 \in (0, \infty)$, we will use the following stopping criteria for terminating Algorithm SNCG:

(A1) $\Psi_l(Y^{l+1}) \leq \Psi_l(Y^l) - \frac{\xi_1}{2} |\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle|$,
(A2) $\|Y^{l+1} - \Pi_{\mathcal{C}}(Y^{l+1} - \nabla \Psi_l(Y^{l+1}))\| \leq \xi_2 (\Psi_l(Y^l) - \Psi_l(Y^{l+1}))^{\frac{1}{2}}$.

We can now state our majorized semismooth Newton-CG method for solving (12) as follows:

Algorithm MSNCG: A Majorized Semismooth Newton-CG Algorithm (MSNCG$(y^0, S^0, Z^0, \widetilde{X}, \sigma)$).

Given $\xi_1 \in (0, 1)$, $\xi_2 \in (0, +\infty)$. Perform the lth iteration as follows.

Step 1. Starting with y^l as the initial point, apply Algorithm SNCG to minimize $\varphi_l(\cdot)$ to find $y^{l+1} = \text{SNCG}(y^l, \widetilde{X}, \sigma)$ and $S^{l+1} := \Pi_{\mathcal{K}^*}(C - \mathcal{A}^* y^{l+1} - Z^l - \sigma^{-1}\widetilde{X})$ satisfying (A1) and (A2).

Step 2. Compute $Z^{l+1} := \Pi_{\mathcal{P}^*}(C - \mathcal{A}^* y^{l+1} - S^{l+1} - \sigma^{-1}\widetilde{X})$.

Next, we establish the convergence of Algorithm MSNCG.

Lemma 2.4 *Suppose that Assumption 2 holds. Then for Algorithm MSNCG,* (A1) *and* (A2) *are achievable.*

Proof If $Y^l - \Pi_{\mathcal{C}}(Y^l - \nabla \Psi_l(Y^l)) = 0$, then one can take $Y^{l+1} = Y^l$ to satisfy (A1) and (A2). Next, we assume that $Y^l - \Pi_{\mathcal{C}}(Y^l - \nabla \Psi_l(Y^l)) \neq 0$. Then Y^l is not an optimal solution of problem (31). Let $\widehat{Y}$ be an arbitrary optimal solution of problem (31). Then $\widehat{Y} = \Pi_{\mathcal{C}}(\widehat{Y} - \nabla \Psi_l(\widehat{Y}))$. So $\langle Y^l - \widehat{Y},\ (\widehat{Y} - \nabla \Psi_l(\widehat{Y})) - \widehat{Y} \rangle \leq 0$, i.e., $\langle \nabla \Psi_l(\widehat{Y}),\ Y^l - \widehat{Y} \rangle \geq 0$, which implies

$$\langle \nabla \Psi_l(Y^l),\ Y^l - \widehat{Y} \rangle \geq \langle \nabla \Psi_l(Y^l) - \nabla \Psi_l(\widehat{Y}),\ Y^l - \widehat{Y} \rangle = \sigma \| \mathcal{M}(\widehat{Y} - Y^l) \|^2. \tag{32}$$

Since

$$\Psi_l(Y^l) > \Psi_l(\widehat{Y}) = \Psi_l(Y^l) + \langle \nabla \Psi_l(Y^l),\ \widehat{Y} - Y^l \rangle + \frac{\sigma}{2} \| \mathcal{M}(\widehat{Y} - Y^l) \|^2, \tag{33}$$

we obtain that $\langle \nabla \Psi_l(Y^l),\ \widehat{Y} - Y^l \rangle + \frac{\sigma}{2}\|\mathcal{M}(\widehat{Y} - Y^l)\|^2 < 0$. This implies

$$\langle \nabla \Psi_l(Y^l),\ \widehat{Y} - Y^l \rangle < 0. \tag{34}$$

Then by using (32), (33), (34) and the fact that $\widehat{Y} = \Pi_{\mathcal{C}}(\widehat{Y} - \nabla \Psi_l(\widehat{Y}))$, we know that for given $\xi_1 \in (0, 1)$ and $\xi_2 \in (0, \infty)$, there exists $\delta > 0$ such that

$$\Psi_l(Y) \leq \Psi_l(Y^l) - \frac{\xi_1}{2}|\langle \nabla \Psi_l(Y^l),\ Y - Y^l \rangle|,$$
$$\|Y - \Pi_{\mathcal{C}}(Y - \nabla \Psi_l(Y))\| \leq \xi_2(\Psi_l(Y^l) - \Psi_l(Y))^{\frac{1}{2}},$$

for all $Y \in \mathcal{C}$ satisfying $\|Y - \widehat{Y}\| < \delta$. Let $\{\tilde{y}^j\}_{j=0}^{+\infty}$ be the sequence generated by SNCG$(\tilde{y}^0, \widetilde{X}, \sigma)$ with $\tilde{y}^0 := y^l$. For each $j \geq 0$, let $\widetilde{S}^j := \Pi_{\mathcal{K}^*}(C - \mathcal{A}^*\tilde{y}^j - Z^l - \sigma^{-1}\widetilde{X})$. Then by Theorem 2.2, we know that $\{(\tilde{y}^j, \widetilde{S}^j)\}$ is a bounded sequence and any accumulation point of $\{(\tilde{y}^j, \widetilde{S}^j)\}$, say $\widehat{Y} := (\hat{y}, \widehat{S})$, is an optimal solution to problem (31). Thus there exists a sufficiently large j such that $Y^{l+1} := (\tilde{y}^j, \widetilde{S}^j)$ satisfying (A1) and (A2). □

Theorem 2.5 *Suppose that Assumption 2 holds. Let Algorithm MSNCG be executed with stopping criteria* (A1) *and* (A2)*. Then it generates a bounded sequence* $\{(y^l, S^l, Z^l)\}$ *and any accumulation point* $(\hat{y}, \widehat{S})$ *of* $\{(y^l, S^l)\}$ *is an optimal solution to problem* (12) *and hence* $(\hat{y}, \widehat{S}, \widehat{Z})$ *is an optimal solution to problem* (8)*, where* $\widehat{Z} := \Pi_{\mathcal{P}^*}(C - \mathcal{A}^*\hat{y} - \widehat{S} - \sigma^{-1}\widetilde{X})$*. Furthermore,* $\|Z^{l+1} - Z^l\| \to 0$ *as* $l \to +\infty$.

Proof By (A1), we have $\Phi(Y^{l+1}) \leq \Psi_l(Y^{l+1}) \leq \Psi_l(Y^l) - \frac{\xi_1}{2}|\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle| = \Phi(Y^l) - \frac{\xi_1}{2}|\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle|$. Hence, the sequence $\{\Phi(Y^l)\}$ is nonincreasing.

By Lemma 2.1, we know that the level set $\mathcal{L} := \{Y \in \mathcal{C} \mid \Phi(Y) \leq \Phi(Y^0)\}$ is a closed and bounded convex set. Then the sequence $\{Y^l\}$ is bounded and so is the sequence $\{Z^l\}$. Let $\widehat{Y}$ be an accumulation point of $\{Y^l\}$. Then $\Phi(Y^l) \to \Phi(\widehat{Y})$ and $\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle \to 0$ as $l \to \infty$. Furthermore, $\Psi_l(Y^l) - \Psi_l(Y^{l+1}) \to 0$ as $l \to \infty$.

By noting that

$$\Psi_l(Y^{l+1}) = \Psi_l(Y^l) + \langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle + \frac{\sigma}{2}\|\mathcal{M}(Y^{l+1} - Y^l)\|^2, \tag{35}$$

we get from (A1) that

$$\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle + \frac{\sigma}{2}\|\mathcal{M}(Y^{l+1} - Y^l)\|^2 \leq -\frac{\xi_1}{2}|\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle| \leq 0. \tag{36}$$

Since $\langle \nabla \Psi_l(Y^l),\ Y^{l+1} - Y^l \rangle \to 0$ as $l \to \infty$, we obtain from (36) that

$$\|\mathcal{M}(Y^{l+1} - Y^l)\| \to 0 \quad \text{as} \quad l \to \infty. \tag{37}$$

For any $l \geq 0$, denote $\Delta_l := Y^l - \Pi_{\mathcal{C}}(Y^l - \nabla \Phi(Y^l))$. Then we have

$$\begin{aligned}\|\Delta_{l+1}\| &\leq \|Y^{l+1} - \Pi_{\mathcal{C}}(Y^{l+1} - \nabla \Psi_l(Y^{l+1}))\| \\ &\quad + \|\Pi_{\mathcal{C}}(Y^{l+1} - \nabla \Psi_l(Y^{l+1})) - \Pi_{\mathcal{C}}(Y^{l+1} - \nabla \Phi(Y^{l+1}))\| \\ &\leq \xi_2(\Psi_l(Y^l) - \Psi_l(Y^{l+1}))^{\frac{1}{2}} + \|\nabla \Psi_l(Y^{l+1}) - \nabla \Phi(Y^{l+1})\|.\end{aligned}$$

By direct computations, we have for $l \geq 1$,

$$\begin{aligned}&\|\nabla \Psi_l(Y^{l+1}) - \nabla \Phi(Y^{l+1})\| \\ &= \|\sigma \mathcal{M}^*(\mathcal{M}Y^{l+1} + \sigma^{-1}\widetilde{X} - C + Z^l) - \sigma \mathcal{M}^*(\Pi_{\mathcal{P}}(\mathcal{M}Y^{l+1} + \sigma^{-1}\widetilde{X} - C))\| \\ &= \|\sigma \mathcal{M}^*(Z^l - Z^{l+1})\| \leq \sigma \|\mathcal{M}^*\| \|\mathcal{M}(Y^{l+1} - Y^l)\|,\end{aligned}$$

where we have used the fact that

$$\begin{aligned}\|Z^{l+1} - Z^l\| &= \|\Pi_{\mathcal{P}^*}(C - \sigma^{-1}\widetilde{X} - \mathcal{M}Y^{l+1}) - \Pi_{\mathcal{P}^*}(C - \sigma^{-1}\widetilde{X} - \mathcal{M}Y^l)\| \\ &\leq \|\mathcal{M}(Y^{l+1} - Y^l)\|. \end{aligned} \tag{38}$$

Thus, by (37) and the fact that $(\Psi_l(Y^l) - \Psi_l(Y^{l+1}))^{\frac{1}{2}} \to 0$ as $l \to \infty$, we derive that $\|\Delta_{l+1}\| \to 0$ as $l \to \infty$. Since $\widehat{Y}$ is an accumulation point of $\{Y^l\}$, we obtain that $\widehat{Y} - \Pi_{\mathcal{C}}(\widehat{Y} - \nabla \Phi(\widehat{Y})) = 0$. By the convexity of Φ, $\widehat{Y}$ is an optimal solution of problem (30).

Finally, by using (37) and (38), we know that $\|Z^{l+1} - Z^l\| \to 0$ as $l \to \infty$. □

3 A majorized semismooth Newton-CG augmented Lagrangian method

For any $k \geq 0$ and $(y, S, Z) \in \Re^m \times \mathcal{S}^n \times \mathcal{S}^n$, denote

$$\phi_k(y, S, Z) := L_{\sigma_k}(y, S, Z; X^k), \tag{39}$$

$$\hat{\phi}_k(y, S, Z) := \begin{cases} L_{\sigma_k}(y, S, Z; X^k) & \text{if } (y, S, Z) \in \Omega := \Re^m \times \mathcal{K}^* \times \mathcal{P}^*, \\ +\infty & \text{otherwise.} \end{cases} \tag{40}$$

Since the inner problems in (8) are solved inexactly, we will use the following standard stopping criteria considered in [18,19] to terminate Algorithm MSNCG:

(B1) $\hat{\phi}_k(y^{k+1}, S^{k+1}, Z^{k+1}) - \inf \hat{\phi}_k \leq \epsilon_k^2, \quad \epsilon_k \geq 0, \quad \sum_{k=0}^{\infty} \epsilon_k < \infty.$
(B2) $\hat{\phi}_k(y^{k+1}, S^{k+1}, Z^{k+1}) - \inf \hat{\phi}_k \leq (\delta_k^2/2\sigma_k)\|X^{k+1} - X^k\|, \quad \delta_k \geq 0, \sum_{k=0}^{\infty} \delta_k < \infty.$
(B3) $\mathrm{dist}(0, \partial \hat{\phi}_k(y^{k+1}, S^{k+1}, Z^{k+1})) \leq (\delta_k'/\sigma_k)\|X^{k+1} - X^k\|, \quad 0 \leq \delta_k' \to 0.$

Just like SDPNAL, each iteration of the MSNCG algorithm can be quite expensive. Thus it is crucial for us to find a reasonably good initial point to warm start Algorithm SDPNAL+. We can certainly do so by solving the inner problem (8) by using any gradient descent type method. However, for this purpose we find that ADMM+

introduced by Sun et al. [22] is usually more efficient than other choices. Now we can present our SDPNAL+ algorithm as follows.

Algorithm SDPNAL+: A Majorized Semismooth Newton-CG Augmented Lagrangian Algorithm (SDPNAL+$(y^0, S^0, Z^0, X^0, \sigma_0)$)

Stage 1. Use ADMM+ to generate an initial point

$$(y^0, S^0, Z^0, X^0, \sigma_0) \leftarrow \text{ADMM+}(y^0, S^0, Z^0, X^0, \sigma_0).$$

Stage 2. For $k = 0, \ldots,$ perform the kth iteration as follows:

(a) Using (y^k, S^k, Z^k) as the initial point, apply Algorithm MSNCG to minimize $\hat{\phi}_k(\cdot)$ to find $(y^{k+1}, S^{k+1}, Z^{k+1}) = \text{MSNCG}(y^k, S^k, Z^k, X^k, \sigma_k)$ and $X^{k+1} = X^k + \sigma_k(\mathcal{A}^* y^{k+1} + S^{k+1} + Z^{k+1} - C)$ satisfying (B1), (B2) or (B3).

(b) Update $\sigma_{k+1} = \rho\sigma_k$ for some $\rho > 1$ or $\sigma_{k+1} = \sigma_k$.

Remark 3.1 (a) As mentioned in the introduction, if (P) is reformulated as a standard SDP and Algorithm SDPNAL+ is applied to this reformulated form, then SDPNAL+ reduces to SDPNAL proposed in [26].

(b) Note that numerically it is difficult to compute $\text{dist}(0, \partial\hat{\phi}_k(W^{k+1}))$ in the criterion (B3) for terminating Algorithm MSNCG directly, where $W^{k+1} = (y^{k+1}, S^{k+1}, Z^{k+1})$. Fortunately, we have from [18] that

$$\begin{aligned}
(\text{dist}(0, \partial\hat{\phi}_k(W^{k+1})))^2 &= \|\Pi_{\mathcal{T}_\Omega(W^{k+1})}(-\nabla\phi_k(W^{k+1}))\|^2 \\
&= \|\Pi_{\mathcal{T}_{\Re^m}(y^{k+1})}(-\nabla_y\phi_k(W^{k+1}))\|^2 \\
&\quad +\|\Pi_{\mathcal{T}_{\mathcal{K}^*}(S^{k+1})}(-\nabla_S\phi_k(W^{k+1}))\|^2 \\
&\quad +\|\Pi_{\mathcal{T}_{\mathcal{P}^*}(Z^{k+1})}(-\nabla_Z\phi_k(W^{k+1}))\|^2 \\
&= \|\mathcal{A}(X^k + \sigma_k R_D^{k+1}) - b\|^2 \\
&\quad +\|\Pi_{\mathcal{T}_{\mathcal{K}^*}(S^{k+1})}(-\sigma_k R_D^{k+1} - X^k)\|^2 \\
&\quad +\|\Pi_{\mathcal{T}_{\mathcal{P}^*}(Z^{k+1})}(-\sigma_k R_D^{k+1} - X^k)\|^2
\end{aligned}$$

where $R_D^{k+1} = \mathcal{A}^* y^{k+1} + S^{k+1} + Z^{k+1} - C$. Observe that the first term in the last equality can readily be evaluated. The third term can also be computed easily since $\mathcal{P}$ is a polyhedral cone. The second term is again computable as we shall show next. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of S^{k+1} being arranged in a nonincreasing order. Denote $\alpha := \{i \mid \lambda_i > 0,\ i = 1, \ldots, n\}$ and $\bar{\alpha} := \{1, \ldots, n\}\backslash\alpha$. Then there exists an orthogonal matrix $P \in \Re^{n\times n}$ such that

$$S^{k+1} = P \begin{bmatrix} \Lambda_\alpha & 0 \\ 0 & 0 \end{bmatrix} P^{\mathrm{T}},$$

where Λ_α is the diagonal matrix whose diagonal entries are λ_i for $i \in \alpha$. Write $P = [P_\alpha\ P_{\bar{\alpha}}]$ with $P_\alpha \in \Re^{n\times|\alpha|}$ and $P_{\bar{\alpha}} \in \Re^{n\times|\bar{\alpha}|}$. From [1], we know that the tangent cone of $\mathcal{S}_+^n$ at $S^{k+1} \in \mathcal{S}_+^n$ can be characterized as $\mathcal{T}_{\mathcal{S}_+^n}(S^{k+1}) = \{B \in \mathcal{S}^n \mid P_{\bar{\alpha}}^T B P_{\bar{\alpha}} \succeq 0\}$. Let $H = P^T(-\sigma_k R_D^{k+1} - X^k)P$. Then

$$\Pi_{\mathcal{T}_{\mathcal{K}^*}(S^{k+1})}(-\sigma_k R_D^{k+1} - X^k) = P \begin{pmatrix} H_{\alpha\alpha} & H_{\alpha\bar{\alpha}} \\ H_{\bar{\alpha}\alpha} & \Pi_{\mathcal{S}_+^{|\bar{\alpha}|}}(H_{\bar{\alpha}\bar{\alpha}}) \end{pmatrix} P^T.$$

We can obtain similar theorems on the convergence of SDPNAL+ as SDPNAL ([26, Theorems 4.1 and 4.2]). The global convergence of Algorithm SDPNAL+ follows from Rockafellar [19, Theorem 1] and [18, Theorem 4] without much difficulty.

Theorem 3.1 *Suppose that Assumption 2 holds. Let Algorithm SDPNAL+ be executed with stopping criterion* (B1)*. If there exists* $(y_0, S_0, Z_0) \in \Re^m \times \mathcal{S}_+^n \times \mathcal{S}^n$ *such that*

$$\mathcal{A}^*(y_0) + S_0 + Z_0 = C, \quad S_0 \succ 0, \quad Z_0 \in relint(\mathcal{P}^*), \tag{41}$$

then the sequence $\{X^k\} \subset \mathcal{P}$ *generated by Algorithm SDPNAL+ is bounded and* $\{X^k\}$ *converges to* $\overline{X}$*, where* $\overline{X}$ *is some optimal solution to* (P)*, and* $\{(y^k, S^k, Z^k)\}$ *is asymptotically minimizing for* (D) *with* $\max(\mathrm{P}) = \inf(\mathrm{D})$.

If $\{X^k\}$ *is bounded, then the sequence* $\{(y^k, S^k, Z^k)\}$ *is also bounded, and all of its accumulation points of the sequence* $\{(y^k, S^k, Z^k)\}$ *are optimal solutions to* (D).

Next we state the local linear convergence of Algorithm SDPNAL+.

Theorem 3.2 *Suppose that Assumption 2 holds. Let Algorithm SDPNAL+ be executed with stopping criteria* (B1) *and* (B2)*. Assume that* (D) *satisfies condition* (41)*. If the second order sufficient conditions* (*in the sense of the conditions in* [2, Theorem 3.137]) *holds at* $\overline{X}$*, where* $\overline{X}$ *is an optimal solution to* (P)*, then the generated sequence* $\{X^k\} \subset \mathcal{P}$ *is bounded and* $\{X^k\}$ *converges to the unique optimal solution* $\overline{X}$ *with* $\max(\mathrm{P}) = \min(\mathrm{D})$*, and*

$$\|X^{k+1} - \overline{X}\| \leq \theta_\infty \|X^k - \overline{X}\| \quad \forall\, k \textit{ sufficiently large,}$$

for some $\theta_\infty \in [0, 1)$ *with the property that* $\theta_\infty \ll 1$ *if* $\sigma_k \to \sigma_\infty$ *for any sufficiently large* σ_∞*. The conclusions of Theorem 3.1 about* $\{(y^k, S^k, Z^k)\}$ *are also valid.*

Proof The conclusions of Theorem 3.2 follow from the results in [19, Theorem 2] and [18, Theorem 5 and Proposition 3] combined with [2, Theorem 3.137]. □

4 Numerical experiments

4.1 SDP+ and SDP problem sets

In our numerical experiments, we test the following SDP+ and SDP problem sets.

(i) SDP+ problems coming from the relaxation of a binary integer nonconvex quadratic (BIQ) programming:

$$\min \left\{ \frac{1}{2} x^T Q x + \langle c, \ x\rangle \mid x \in \{0, 1\}^{n-1} \right\}. \tag{42}$$

This problem has been shown in [4] that under some mild assumptions, it can equivalently be reformulated as the following completely positive programming (CPP) problem:

$$\min\left\{\frac{1}{2}\langle Q,\ X_0\rangle + \langle c,\ x\rangle \mid \operatorname{diag}(X_0) = x, \quad X = [X_0, x; x^T, 1] \in \mathcal{C}^n_{pp}\right\}, \quad (43)$$

where $\mathcal{C}^n_{pp}$ denotes the n-dimensional completely positive cone. It is well known that even though $\mathcal{C}^n_{pp}$ is convex, it is computationally intractable. To solve the CPP problem, one would typically relax $\mathcal{C}^n_{pp}$ to $\mathcal{S}^n_+ \cap \mathcal{S}^n_{\geq 0}$, and the relaxed problem has the form (P):

$$\begin{array}{ll} \min & \frac{1}{2}\langle Q,\ X_0\rangle + \langle c,\ x\rangle \\ \text{s.t.} & \operatorname{diag}(X_0) - x = 0, \quad \alpha = 1, \quad X = \begin{bmatrix} X_0 & x \\ x^T & \alpha \end{bmatrix} \in \mathcal{S}^n_+, \quad X \in \mathcal{P}, \end{array} \quad (44)$$

where the polyhedral cone $\mathcal{P} = \{X \in \mathcal{S}^n \mid X \geq 0\}$. In our numerical experiments, the test data for Q and c are taken from Biq Mac Library maintained by Wiegele, which is available at http://biqmac.uni-klu.ac.at/biqmaclib.html.

(ii) SDP and SDP+ problems arising from the relaxation of maximum stable set problems. Given a graph G with edge set $\mathcal{E}$, the SDP and SDP+ relaxation $\theta(G)$ and $\theta_+(G)$ of the maximum stable set problem are given by

$$\theta(G) = \max\{\langle ee^T,\ X\rangle : \langle E_{ij},\ X\rangle = 0, \quad (i,j) \in \mathcal{E}, \quad \langle I,\ X\rangle = 1, \quad X \in \mathcal{S}^n_+\}, \quad (45)$$

$$\theta_+(G) = \max\{\langle ee^T,\ X\rangle : \langle E_{ij},\ X\rangle = 0, \quad (i,j) \in \mathcal{E}, \quad \langle I,\ X\rangle = 1, \quad X \in \mathcal{S}^n_+, \quad X \in \mathcal{P}\}, \quad (46)$$

where $E_{ij} = e_i e_j^{\mathrm{T}} + e_j e_i^{\mathrm{T}}$ and e_i denotes the ith column of the identity matrix, $\mathcal{P} = \{X \in \mathcal{S}^n \mid X \geq 0\}$. In our numerical experiments, we test the graph instances G considered in [20,23,24].

(iii) SDP+ relaxation for computing lower bounds for quadratic assignment problems (QAPs). Let Π be the set of $n \times n$ permutation matrices. Given matrices $A, B \in \mathcal{S}^n$, the QAP is given by

$$v^*_{\mathrm{QAP}} := \min\{\langle X, AXB\rangle : X \in \Pi\}. \quad (47)$$

For a matrix $X = [x_1, \ldots, x_n] \in \Re^{n\times n}$, we will identify it with the n^2-vector $x = [x_1; \ldots; x_n]$. For a matrix $Y \in R^{n^2\times n^2}$, we let Y^{ij} be the $n \times n$ block corresponding to $x_i x_j^T$ in the matrix xx^T. It is shown in [16] that v^*_{QAP} is bounded below by the following number generated from the SDP+ relaxation of (47):

$$\begin{array}{rl} v := \min & \langle B \otimes A,\ Y\rangle \\ \text{s.t.} & \sum_{i=1}^n Y^{ii} = I,\ \langle I,\ Y^{ij}\rangle = \delta_{ij} \quad \forall\, 1 \leq i \leq j \leq n, \\ & \langle E,\ Y^{ij}\rangle = 1 \quad \forall\, 1 \leq i \leq j \leq n, \quad Y \in \mathcal{S}^n_+,\ Y \in \mathcal{P}, \end{array} \quad (48)$$

where the sign $\otimes$ stands for the Kronecker product, E is the matrix of ones, and $\delta_{ij} = 1$ if $i = j$, and 0 otherwise, $\mathcal{P} = \{X \in \mathcal{S}^{n^2} \mid X \geq 0\}$. In our numerical experiments, the test instances (A, B) are taken from the QAP Library [8].

(iv) SDP+ relaxations of clustering problems (RCPs) described in [15, eq. (13), uptoaconstant]:

$$\min\{\langle -W,\ X\rangle \mid Xe = e, \langle I,\ X\rangle = K, \quad X \in \mathcal{S}^n_+, \quad X \in \mathcal{P}\}, \tag{49}$$

where W is the so-called affinity matrix whose entries represent the similarities of the objects in the dataset, e is the vector of ones, and K is the number of clusters, $\mathcal{P} = \{X \in \mathcal{S}^n \mid X \geq 0\}$. All the data sets we tested are from the UCI Machine Learning Repository (available at http://archive.ics.uci.edu/ml/datasets.html). For some large data instances, we only select the first n rows. For example, the original data instance "`spambase`" has 4601 rows, we select the first 1500 rows to obtain the test problem "`spambase-large.2`" for which the number "2" means that there are $K = 2$ clusters.

(v) SDP+ problems arising from the SDP relaxation of frequency assignment problems (FAPs) [7]. Given a network represented by a graph G and an edge-weight matrix W, a certain type of frequency assignment problem on G can be relaxed into the following SDP (see [5, eq. (5)]):

$$\begin{array}{ll} \max & \left\langle \left(\frac{k-1}{2k}\right) L(G, W) - \frac{1}{2}\mathrm{Diag}(We)X \right\rangle \\ \text{s.t.} & \mathrm{diag}(X) = e, X \in \mathcal{S}^n_+, \\ & -E^{ij} \bullet X = 2/(k-1) \quad \forall (i, j) \in U \subseteq E, \\ & -E^{ij} \bullet X \leq 2/(k-1) \quad \forall (i, j) \in E \backslash U, \end{array} \tag{50}$$

where $k > 1$ is an integer, $L(G, W) := \mathrm{Diag}(We) - W$ is the Laplacian matrix, $E^{ij} = e_i e_j^T + e_j e_i^T$ with $e_i \in \Re^n$ being the ith standard unit vector and $e \in \Re^n$ is the vector of all ones. Define $M_{ij} = -\frac{1}{k-1}$ if $(i, j) \in E$, and $M_{ij} = 0$ otherwise. Then (50) is equivalent to

$$\begin{array}{ll} \max & \left\langle \left(\frac{k-1}{2k}\right) L(G, W) - \frac{1}{2}\mathrm{Diag}(We)X \right\rangle \\ \text{s.t.} & \mathrm{diag}(X) = e, X \in \mathcal{S}^n_+, X - M \in \mathcal{P}, \end{array} \tag{51}$$

where $\mathcal{P} = \{X \in \mathcal{S}^n \mid X_{ij} = 0, \quad \forall (i, j) \in U; \quad X_{ij} \geq 0, \quad \forall (i, j) \in E \backslash U\}$.

We should mention that we can easily extend our algorithm to handle the following more general SDP+ problem:

$$\min\{\langle C,\ X\rangle \mid \mathcal{A}(X) = b, \quad X \in \mathcal{K}, \quad X - M \in \mathcal{P}\}, \tag{52}$$

where $M \in \mathcal{X}$ is a given matrix. Thus (51) can also be solved by our proposed algorithm.

(vi) SDP relaxations for rank-1 tensor approximations (R1TA) [13]:

$$\max\{\langle f,\ y\rangle \mid M(y) \in \mathcal{S}^n_+, \quad \langle g,\ y\rangle = 1\}, \tag{53}$$

where $y \in \Re^{\mathbb{N}^n_m}$, $M(y)$ is a linear pencil in y. The dual of (53) is given by

$$\min\{\gamma \mid \gamma g - f = M^*(X), \quad X \in \mathcal{S}^n_+\}. \tag{54}$$

It is shown in [12] that (54) can be transformed into a standard SDP (up to a constant):

$$\min\{\langle C, \ X\rangle \mid \mathcal{A}(X) = b, \quad X \in \mathcal{S}^n_+\}, \tag{55}$$

where C is a constant matrix and $\mathcal{A}$ is a linear map, which depend on M, f, g.

4.2 Numerical results

In this subsection, we compare the performance of our SDPNAL+ algorithm with two other competitive publicly available first order methods based codes for solving large-scale SDP+ and SDP problems: an ADMM based solver, called SDPAD (release-beta2, released in December 2012) developed in [25] and a two-easy-block-decomposition hybrid proximal extragradient method, which was called 2EBD-HPE[1] (v0.2, released on May 31, 2013) and we call it 2EBD here, introduced in [10]. Since we use the convergent ADMM with 3-block constraints introduced by Sun et al. [22] (which was called ADMM3c but we call it ADMM+ here to indicate that it is an enhanced version of ADMM with convergence guarantee) to warm start SDPNAL+, we also list the numerical results obtained by running ADMM+ alone for the purpose of demonstrating the power and the importance of the proposed majorized semismooth Newton-CG algorithm for solving difficult SDP+ and SDP problems.

All our computational results for the tested SDP+ and SDP problems are obtained by running MATLAB on a Linux server (6-core, Intel Xeon X5650 @ 2.67 GHz, 32 G RAM).

In our numerical experiments, we measure the accuracy of an approximate optimal solution (X, y, S, Z) for (P) and (D) by using the following relative residual:

$$\eta = \max\{\eta_P, \eta_D, \eta_{\mathcal{K}}, \eta_{\mathcal{P}}, \eta_{\mathcal{K}^*}, \eta_{\mathcal{P}^*}, \eta_{C1}, \eta_{C2}\}, \tag{56}$$

where $\eta_P = \frac{\|\mathcal{A}X-b\|}{1+\|b\|}$, $\eta_D = \frac{\|\mathcal{A}^*y+S+Z-C\|}{1+\|C\|}$, $\eta_{\mathcal{K}} = \frac{\|\Pi_{\mathcal{K}^*}(-X)\|}{1+\|X\|}$, $\eta_{\mathcal{P}} = \frac{\|\Pi_{\mathcal{P}^*}(-X)\|}{1+\|X\|}$, $\eta_{\mathcal{K}^*} = \frac{\|\Pi_{\mathcal{K}}(-S)\|}{1+\|S\|}$, $\eta_{\mathcal{P}^*} = \frac{\|\Pi_{\mathcal{P}}(-Z)\|}{1+\|Z\|}$, $\eta_{C1} = \frac{|\langle X, \ S\rangle|}{1+\|X\|+\|S\|}$, $\eta_{C2} = \frac{|\langle X, \ Z\rangle|}{1+\|X\|+\|Z\|}$. Additionally, we compute the relative gap by

$$\eta_g = \frac{\langle C, \ X\rangle - \langle b, \ y\rangle}{1 + |\langle C, \ X\rangle| + |\langle b, \ y\rangle|}. \tag{57}$$

Let $\varepsilon > 0$ be a given accuracy tolerance. We terminate both SDPNAL+ and ADMM+ when $\eta < \varepsilon$.

1 http://www2.isye.gatech.edu/~cod3/CamiloOrtiz/Software_files/2EBD-HPE_v0.2/2EBD-HPE_v0.2.zip

Note that SDPAD can be used to solve SDP+ problems of form (P) with $\mathcal{P} = \mathcal{S}^n_{\geq 0}$ directly and we stop SDPAD when $\eta < \varepsilon$, where η is defined as in (56). However, it is shown recently that the direct extension of ADMM to the multi-block case is not necessarily convergent [6]. Hence SDPAD, which is essentially an implementation of the direct extension of ADMM with the step length set at 1.618 for solving the dual of SDP+ problems, does not have convergence guarantee in theory.

The implementation of 2EBD including its termination, along with ADMM+ and SDPAD, is done in the same way as in [22]. For 2EBD, we reformulate QAP, RCP and R1TA problems as SDP problems in the standard form as these problems do not appear to have obvious two-easy blocks structures.

In our numerical experiments, we also use a restart strategy for SDPNAL+ if it is not able to achieve the required accuracy for the tested SDP+ problems. For some problems, even though η_P and η_D can reach the required accuracy tolerance, $\eta_{\mathcal{K}}$ or η_{C1} may stay above the required tolerance or stagnate. This may happen, as in the case for SDPNAL, because many of these SDP+ problems are degenerate at the optimal solutions. One way to overcome this difficulty is to apply ADMM+ to (P) using the most recently computed (y, S, Z, X, σ) to restart SDPNAL+ when its progress is not satisfactory. From this point of view, our proposed algorithm is quite flexible. In addition, the penalty parameter σ is dynamically adjusted according to the progress of the algorithm. A greater σ ensures faster convergence in theory but it leads to a more difficult inner problem (7a). Hence we adjust σ in order to balance this dilemma. However, the exact details on the restart and adjustment strategies are too tedious to be presented here. Note that MSNCG can be viewed as an alternating minimization between the blocks (y, S) and Z for solving problem (7a). Naturally we can try to alternately minimize the blocks (y, Z) and S, for which the computational cost of the generalized Newton system is cheaper due to the simple structure of $\mathcal{P}$. However, the overall cost can be much more expensive because the piecewise linear structure of $\Pi_{\mathcal{P}}(\cdot)$ generally does not give rise to a well-conditioned generalized Hessian, which leads to a slower convergence for solving the problem (7a).

Table 1 shows the number of problems that have been successfully solved to the accuracy of 10^{-6} in η by each of the four solvers SDPNAL+, ADMM+, SDPAD and 2EBD, with the maximum number of iterations set at 25,000 or the maximum

Table 1 Number of problems which are solved to the accuracy of 10^{-6} in η

Problem set (no.)\solver	SDPNAL+	ADMM+	SDPAD	2EBD
θ (58)	58	56	53	53
θ_+ (58)	58	58	58	56
FAP (7)	7	7	7	7
QAP (95)	95	39	30	16
BIQ (134)	134	134	134	134
RCP (120)	120	120	114	109
R1TA (55)	55	42	47	18
Total (527)	527	456	443	393

computation time set at 99 h. As can be seen, only SDPNAL+ can solve all the problems to the accuracy of 10^{-6}. In particular, for the first time, we are able to solve all the 95 difficult SDP+ problems arising from QAP problems to an accuracy of 10^{-6} efficiently, while ADMM+, SDPAD and 2EBD can successfully solve 39, 30 and 16 problems, respectively.

Tables 2 and 3 show the numerical results obtained by SDPNAL+ with the tolerance $\varepsilon = 10^{-6}$ for a subset of of the tested problems (the full set of numerical results can be found at http://www.math.nus.edu.sg/~mattohkc/publist.html/). The first three columns of each table give the problem name, the dimension of the variable y (m), the size of the matrix C (n_s) and the number of linear inequality constraints (n_l) in (D), respectively. The middle five columns give the number of outer iterations, the total number of inner iterations, the total number of iterations for ADMM+, and the objective values $\langle C, \ X\rangle$ and $\langle b, \ y\rangle$. The relative infeasibilities and gap, as well as times (in the format hours:minutes:seconds) are listed in the last eight columns. It is interesting to note that all the tested problems (especially the QAPs) can be solved to the required accuracy 10^{-6} efficiently by SDPNAL+.

Tables 4 and 5 compare SDPNAL+, ADMM+, SDPAD and 2EBD on a subset of the tested SDP+ and SDP problems, respectively, using the tolerance $\varepsilon = 10^{-6}$. We terminate ADMM+, SDPAD, 2EBD after 25,000 iterations or 99 h. As can be seen, except for SDPNAL+, the required accuracy is not achieved for most of the tested QAPs after 25,000 iterations for the solvers ADMM+, SDPAD and 2EBD. For the last three solvers, they typically converge very slowly when η falls below the range of 10^{-4} to 10^{-5}. For R1TA problems, SDPNAL+ is significantly faster than the other 3 methods and it seems that only SDPNAL+ can solve those large scale ($n \geq 2000$) problems efficiently.

We observe that although ADMM+ and SDPAD perform similar steps in each iteration cycle (except that the former perform one extra update on the variable y to ensure the convergence of the algorithm), the former can be more efficient than latter on many tested instances. The main factors to account for the difference in the performance could be (a) ADMM+ can take a larger step length for updating the multiplier (it is shown in [22] that the step length can be taken to be larger than $(1+\sqrt{5})/2$ when a certain checkable criterion holds); (b) it uses a more effective adjustment strategy for updating the penalty parameter σ; (c) in addition, ADMM+ also performs rescaling of the SDP data and restarting the algorithm whenever the ratio in $\|X^k\|$ and $\max\{\|S^k\|, \|Z^k\|, \|\mathcal{A}^* y^k\|\}$ deviate, say more than 20 % from 1.

Figure 1 shows the performance profiles of SDPNAL+, ADMM+, SDPAD and 2EBD for all the 527 tested problems. We recall that a point (x, y) is in the performance profile curve of a method if and only if it can be solved exactly $(100y)$ % of all the tested problems at most x times slower than any other method. It can be seen that SDPNAL+ outperforms the other 3 methods by a significant margin.

Table 2 Performance of SDPNAL+ on θ_+, FAP, QAP, BIQ and RCP problems ($\varepsilon = 10^{-6}$)

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>theta10 | 12470 | 500;</td><td>0| 0|354</td><td>8.31485880 1 | 8.31490052 1</td><td>4.1-14| 8.5-7| 2.2-10| 6.5-9| 1.7-7| 2.5-8| −2.5-6</td><td>46</td></tr>
<tr><td>theta102 | 37467 | 500;</td><td>0| 0|157</td><td>3.80662264 1 | 3.80662724 1</td><td>1.1-13| 9.5-7| 8.5-9| 7.0-9| 1.7-8| 9.6-10| −6.0-7</td><td>23</td></tr>
<tr><td>theta103 | 62516 | 500;</td><td>0| 0|144</td><td>2.23774188 1 | 2.23774202 1</td><td>1.4-14| 9.2-7| 3.9-8| 1.2-8| 9.5-10| 2.1-10| −3.0-8</td><td>22</td></tr>
<tr><td>theta104 | 87245 | 500;</td><td>0| 0|169</td><td>1.32826073 1 | 1.32826099 1</td><td>1.7-14| 9.3-7| 1.5-8| 2.6-9| 2.1-9| 6.9-10| −9.2-8</td><td>24</td></tr>
<tr><td>theta12 | 17979 | 600;</td><td>0| 0|362</td><td>9.20905068 1 | 9.20909180 1</td><td>8.7-13| 9.0-7| 0 | 6.1-9| 1.2-7| 1.4-8| −2.2-6</td><td>1:15</td></tr>
<tr><td>theta123 | 90020 | 600;</td><td>0| 0|156</td><td>2.44951466 1 | 2.44951496 1</td><td>6.6-14| 9.3-7| 1.3-8| 4.1-9| 1.4-9| 3.8-10| −6.0-8</td><td>34</td></tr>
<tr><td>hamming-9-8 | 2305 | 512;</td><td>11| 11|500</td><td>2.24000000 2 | 2.24000020 2</td><td>1.1-10| 9.5-7| 1.7-10| 0.0-16| 1.4-11| 0.0-16| −4.4-8</td><td>44</td></tr>
<tr><td>hamming-10-2 | 23041 | 1024;</td><td>0| 0|657</td><td>8.53345652 1 | 8.53332571 1</td><td>6.9-12| 8.7-7| 2.1-7| 3.2-8| 2.3-7| 3.5-8| 7.6-6</td><td>3:09</td></tr>
<tr><td>hamming-9-5-6 | 53761 | 512;</td><td>0| 0|461</td><td>5.86651695 1 | 5.86665968 1</td><td>3.3-13| 7.5-7| 0 | 0 | 9.5-7| 2.6-7| −1.2-5</td><td>45</td></tr>
<tr><td>G43 | 9991 | 1000;</td><td>21| 21|973</td><td>2.79738220 2 | 2.79735897 2</td><td>5.3-12| 8.9-7| 1.7-7| 2.7-8| 2.1-7| 1.6-8| 4.1-6</td><td>12:32</td></tr>
<tr><td>G44 | 9991 | 1000;</td><td>21| 21|942</td><td>2.79743352 2 | 2.79746225 2</td><td>5.7-12| 9.9-7| 1.2-8| 1.5-8| 2.6-7| 5.0-8| −5.1-6</td><td>12:00</td></tr>
<tr><td>G45 | 9991 | 1000;</td><td>21| 21|888</td><td>2.79313957 2 | 2.79317590 2</td><td>3.6-12| 9.8-7| 6.0-8| 3.9-8| 3.3-7| 1.1-7| −6.5-6</td><td>11:43</td></tr>
<tr><td>G46 | 9991 | 1000;</td><td>21| 21|887</td><td>2.79026490 2 | 2.79032512 2</td><td>3.6-12| 9.7-7| 5.8-8| 4.8-8| 5.6-7| 1.4-7| −1.1-5</td><td>11:35</td></tr>
<tr><td>G47 | 9991 | 1000;</td><td>21| 21|1042</td><td>2.80888870 2 | 2.80891901 2</td><td>9.7-12| 9.4-7| 7.2-10| 1.5-8| 2.7-7| 2.4-8| −5.4-6</td><td>13:10</td></tr>
<tr><td>G51 | 5910 | 1000;</td><td>1| 2|5672</td><td>3.49000718 2 | 3.49000007 2</td><td>8.6-10| 9.9-7| 4.5-7| 4.9-7| 4.2-8| 1.8-8| 1.0-6</td><td>1:15:12</td></tr>
<tr><td>G52 | 5917 | 1000;</td><td>5| 5|10840</td><td>3.48386577 2 | 3.48386413 2</td><td>3.4-9| 9.9-7| 3.9-7| 2.3-7| 1.5-8| 2.3-8| 2.4-7</td><td>2:21:46</td></tr>
<tr><td>G53 | 5915 | 1000;</td><td>4| 4|13260</td><td>3.48213570 2 | 3.48211536 2</td><td>3.3-9| 9.9-7| 5.0-7| 3.0-7| 1.2-7| 4.9-8| 2.9-6</td><td>2:48:21</td></tr>
<tr><td>G54 | 5917 | 1000;</td><td>8| 8|4278</td><td>3.40999655 2 | 3.41000193 2</td><td>4.0-10| 9.9-7| 1.3-7| 1.3-7| 2.0-8| 9.3-9| −7.9-7</td><td>51:18</td></tr>
<tr><td>1dc.1024 | 24064 | 1024;</td><td>0| 0|2620</td><td>9.55514182 1 | 9.55511660 1</td><td>3.2-13| 9.9-7| 2.8-7| 6.4-8| 3.9-8| 4.0-9| 1.3-6</td><td>31:46</td></tr>
<tr><td>1et.1024 | 9601 | 1024;</td><td>0| 0|1144</td><td>1.82071997 2 | 1.82071515 2</td><td>3.1-12| 9.9-7| 2.5-7| 3.5-8| 4.5-8| 2.2-9| 1.3-6</td><td>12:43</td></tr>
<tr><td>1tc.1024 | 7937 | 1024;</td><td>0| 0|2732</td><td>2.04205928 2 | 2.04204076 2</td><td>7.4-13| 9.9-7| 6.3-7| 5.1-7| 1.5-7| 2.9-8| 4.5-6</td><td>31:34</td></tr>
<tr><td>1zc.1024 | 16641 | 1024;</td><td>0| 0|711</td><td>1.28001293 2 | 1.27999917 2</td><td>3.9-12| 7.7-7| 1.6-7| 2.0-8| 1.8-7| 1.7-8| 5.4-6</td><td>7:12</td></tr>
<tr><td>2dc.1024 | 169163 | 1024;</td><td>0| 0|4135</td><td>1.77104688 1 | 1.77099903 1</td><td>4.6-13| 6.5-7| 6.2-7| 9.9-7| 1.6-7| 1.7-7| 1.3-5</td><td>44:55</td></tr>
<tr><td>1dc.2048 | 58368 | 2048;</td><td>0| 0|4153</td><td>1.74258920 2 | 1.74257466 2</td><td>1.1-11| 9.9-7| 3.7-7| 2.9-7| 9.5-8| 1.2-8| 4.2-6</td><td>5:50:06</td></tr>
</table>

Table 2 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>1et.2048 | 22529 | 2048;</td><td>0| 0|3039</td><td>3.38165943 2 | 3.38165218 2</td><td>1.2-11| 9.9-7| 1.8-7| 1.7-7| 2.7-8| 8.6-9| 1.1-6</td><td>4:01:54</td></tr>
<tr><td>1tc.2048 | 18945 | 2048;</td><td>0| 0|2876</td><td>3.70489820 2 | 3.70488730 2</td><td>1.4-11| 9.9-7| 2.8-7| 1.8-7| 3.9-8| 2.2-9| 1.5-6</td><td>3:50:43</td></tr>
<tr><td>2dc.2048 | 504452 | 2048;</td><td>0| 0|2997</td><td>2.87872690 1 | 2.87867849 1</td><td>1.9-12| 9.9-7| 2.9-7| 6.1-7| 7.3-8| 7.0-8| 8.3-6</td><td>3:54:58</td></tr>
<tr><td>fap11 | 252 | 252;</td><td>5| 5|1180</td><td>2.97984952-2 | 2.98707469-2</td><td>8.1-12| 9.6-7| 4.9-7| 4.5-15| 2.4-7| 6.5-15| −6.8-5</td><td>39</td></tr>
<tr><td>fap12 | 369 | 369;</td><td>15| 15|1768</td><td>2.73312482-1 | 2.73415267-1</td><td>9.4-14| 9.9-7| 2.6-8| 6.0-7| 7.0-9| 4.6-7| −6.6-5</td><td>1:56</td></tr>
<tr><td>fap25 | 2118 | 2118;</td><td>11| 11|2268</td><td>1.28781491 1 | 1.28803442 1</td><td>2.2-7| 9.1-7| 0.8-15| 4.5-7| 4.7-16| 9.2-7| −8.2-5</td><td>3:58:21</td></tr>
<tr><td>fap36 | 4110 | 4110;</td><td>4| 4|2033</td><td>6.98573185 1 | 6.98607976 1</td><td>8.4-10| 9.5-7| 8.5-7| 7.0-15| 2.3-8| 1.1-13| −2.5-5</td><td>23:07:56</td></tr>
<tr><td>bur26a | 1051 | 676;</td><td>137|222|10228</td><td>5.42644954 6 | 5.42664508 6</td><td>7.6-11| 9.9-7| 2.6-7| 7.9-7| 5.1-7| 2.5-9| −1.8-5</td><td>1:48:05</td></tr>
<tr><td>bur26c | 1051 | 676;</td><td>247|441|21498</td><td>5.42685366 6 | 5.42707215 6</td><td>5.3-11| 9.9-7| 1.6-7| 7.5-7| 2.5-7| 2.0-9| −2.0-5</td><td>2:03:12</td></tr>
<tr><td>bur26d | 1051 | 676;</td><td>173|306|13287</td><td>3.82088460 6 | 3.82098028 6</td><td>1.5-10| 9.9-7| 9.4-8| 8.0-7| 2.8-7| 3.5-10| −1.3-5</td><td>1:59:20</td></tr>
<tr><td>bur26e | 1051 | 676;</td><td>129|361|14705</td><td>5.38699884 6 | 5.38711462 6</td><td>2.3-10| 8.8-7| 1.1-7| 9.4-7| 1.4-8| 2.5-8| −1.1-5</td><td>1:18:35</td></tr>
<tr><td>bur26f | 1051 | 676;</td><td>107|248|11272</td><td>3.78211965 6 | 3.78219580 6</td><td>2.1-15| 9.9-7| 4.1-7| 2.7-7| 3.8-11| 4.1-9| −1.0-5</td><td>1:45:13</td></tr>
<tr><td>bur26g | 1051 | 676;</td><td>250|392|10817</td><td>1.01172603 7 | 1.01177521 7</td><td>2.3-11| 9.9-7| 5.0-7| 5.6-7| 2.3-9| 4.2-9| −2.4-5</td><td>1:32:44</td></tr>
<tr><td>bur26h | 1051 | 676;</td><td>146|360|10658</td><td>7.09871739 6 | 7.09844876 6</td><td>1.5-10| 9.5-7| 7.4-7| 9.9-7| 2.4-9| 4.7-9| 1.9-5</td><td>1:25:45</td></tr>
<tr><td>chr22a | 757 | 484;</td><td>104|250|6940</td><td>6.15599997 3 | 6.15591877 3</td><td>7.8-9| 3.5-7| 2.6-10| 0.2-16| 3.5-11| 0.0-16| 6.6-6</td><td>22:57</td></tr>
<tr><td>chr22b | 757 | 484;</td><td>89|189|5620</td><td>6.19399919 3 | 6.19397816 3</td><td>2.0-7| 3.1-7| 1.6-7| 1.9-16| 6.3-10| 0.0-16| 1.7-6</td><td>12:46</td></tr>
<tr><td>chr25a | 973 | 625;</td><td>53|200|5151</td><td>3.79659302 3 | 3.79236524 3</td><td>2.5-11| 8.6-7| 6.5-8| 3.7-8| 4.2-8| 2.0-7| 5.6-4</td><td>21:04</td></tr>
<tr><td>esc32a | 1582 | 1024;</td><td>46| 78|1664</td><td>1.03320457 2 | 1.03320681 2</td><td>3.3-13| 9.5-7| 9.9-7| 8.1-7| 6.6-9| 7.5-9| −1.1-6</td><td>32:32</td></tr>
<tr><td>esc32b | 1582 | 1024;</td><td>52|100|2196</td><td>1.31863123 2 | 1.31876533 2</td><td>7.2-12| 9.8-7| 3.7-7| 9.6-7| 6.7-8| 9.3-7| −5.1-5</td><td>45:50</td></tr>
<tr><td>esc32c | 1582 | 1024;</td><td>46|139|3562</td><td>6.15172917 2 | 6.15177999 2</td><td>3.0-7| 9.7-7| 8.6-13| 3.2-7| 3.3-13| 1.4-7| −4.1-6</td><td>1:06:19</td></tr>
<tr><td>esc32d | 1582 | 1024;</td><td>0| 0|678</td><td>1.90223554 2 | 1.90227139 2</td><td>3.5-12| 9.9-7| 2.2-7| 2.7-7| 3.1-7| 2.3-7| −9.4-6</td><td>9:40</td></tr>
<tr><td>esc32e | 1582 | 1024;</td><td>40| 47|1248</td><td>1.90001450 0 | 1.89997253 0</td><td>1.6-8| 9.9-7| 1.3-15| 1.4-8| 1.8-16| 6.4-9| 8.7-6</td><td>22:00</td></tr>
<tr><td>esc32f | 1582 | 1024;</td><td>40| 47|1248</td><td>1.90001450 0 | 1.89997253 0</td><td>1.6-8| 9.9-7| 1.3-15| 1.4-8| 1.8-16| 6.4-9| 8.7-6</td><td>21:31</td></tr>
<tr><td>esc32g | 1582 | 1024;</td><td>0| 0|520</td><td>5.83333959 0 | 5.83331567 0</td><td>2.5-13| 9.3-7| 9.7-9| 8.0-8| 2.1-10| 4.4-9| 1.9-6</td><td>7:17</td></tr>
</table>

Table 2 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>kra30a | 1393 | 900;</td><td>49| 72|3208</td><td>8.68155154 4 | 8.68268686 4</td><td>1.2-10| 7.5-7| 1.2-7| 9.9-7| 3.4-7| 1.9-7| −6.5-5</td><td>52:13</td></tr>
<tr><td>kra30b | 1393 | 900;</td><td>81|101|3080</td><td>8.78355000 4 | 8.78468655 4</td><td>7.1-11| 7.0-7| 1.4-7| 9.9-7| 3.7-7| 1.7-7| −6.5-5</td><td>55:48</td></tr>
<tr><td>kra32 | 1582 | 1024;</td><td>67| 83|2946</td><td>8.57529179 4 | 8.57648412 4</td><td>1.3-11| 8.4-7| 1.4-7| 9.9-7| 3.5-7| 9.3-7| −7.0-5</td><td>1:07:22</td></tr>
<tr><td>lipa30a | 1393 | 900;</td><td>443|1216|1300</td><td>1.31780000 4 | 1.31780000 4</td><td>4.4-8| 1.0-7| 1.0-8| 1.9-16| 2.5-13| 0.0-16| −4.3-10</td><td>47:33</td></tr>
<tr><td>lipa30b | 1393 | 900;</td><td>4| 9|820</td><td>1.51426000 5 | 1.51426111 5</td><td>2.3-9| 8.2-9| 3.3-14| 4.7-16| 4.1-14| 0.1-16| −3.7-7</td><td>11:08</td></tr>
<tr><td>lipa40a | 2458 | 1600;</td><td>153|546|3732</td><td>3.15379935 4 | 3.15379159 4</td><td>5.5-7| 1.2-7| 7.7-8| 1.4-16| 5.6-10| 0.0-16| 1.2-6</td><td>3:01:05</td></tr>
<tr><td>lipa40b | 2458 | 1600;</td><td>5| 18|991</td><td>4.76581311 5 | 4.76581129 5</td><td>4.2-7| 1.5-7| 3.3-7| 5.2-16| 3.7-9| 0.0-16| 1.9-7</td><td>1:02:59</td></tr>
<tr><td>nug24 | 898 | 576;</td><td>43| 66|2359</td><td>3.40071083 3 | 3.40090464 3</td><td>1.0-10| 9.6-7| 1.1-7| 9.9-7| 2.6-7| 2.5-7| −2.8-5</td><td>14:12</td></tr>
<tr><td>nug25 | 973 | 625;</td><td>48| 76|2708</td><td>3.62566278 3 | 3.62577908 3</td><td>9.7-11| 7.9-7| 6.0-8| 9.9-7| 1.1-7| 7.2-8| −1.6-5</td><td>19:25</td></tr>
<tr><td>nug27 | 1132 | 729;</td><td>49| 86|3300</td><td>5.12924842 3 | 5.12946060 3</td><td>1.5-10| 9.0-7| 6.2-8| 9.9-7| 1.2-7| 3.2-7| −2.1-5</td><td>36:53</td></tr>
<tr><td>nug28 | 1216 | 784;</td><td>50| 77|3190</td><td>5.02532199 3 | 5.02552528 3</td><td>1.6-10| 8.7-7| 7.2-8| 9.9-7| 1.3-7| 4.4-7| −2.0-5</td><td>40:26</td></tr>
<tr><td>nug30 | 1393 | 900;</td><td>44| 68|2463</td><td>5.94890136 3 | 5.94920345 3</td><td>3.8-11| 9.9-7| 1.3-7| 9.6-7| 3.3-7| 1.1-7| −2.5-5</td><td>45:02</td></tr>
<tr><td>ste36a | 1996 | 1296;</td><td>122|189|7344</td><td>9.25661751 3 | 9.25811948 3</td><td>4.3-11| 9.9-7| 1.6-7| 9.9-7| 3.9-7| 6.8-8| −8.1-5</td><td>6:29:21</td></tr>
<tr><td>ste36b | 1996 | 1296;</td><td>173|242|11851</td><td>1.56582438 4 | 1.56653460 4</td><td>1.5-10| 9.9-7| 1.8-7| 9.9-7| 5.4-7| 4.2-8| −2.3-4</td><td>9:45:58</td></tr>
<tr><td>ste36c | 1996 | 1296;</td><td>143|202|10008</td><td>8.13267040 6 | 8.13407147 6</td><td>4.6-11| 9.9-7| 1.8-7| 9.5-7| 4.8-7| 3.8-9| −8.6-5</td><td>8:06:50</td></tr>
<tr><td>tai25a | 973 | 625;</td><td>33| 42|2630</td><td>1.11336476 6 | 1.11525360 6</td><td>8.5-12| 9.9-7| 5.1-8| 9.4-7| 3.0-9| 2.7-9| −8.5-4</td><td>14:52</td></tr>
<tr><td>tai25b | 973 | 625;</td><td>296|344|18325</td><td>3.37687430 8 | 3.37871783 8</td><td>1.7-10| 9.9-7| 1.7-7| 9.9-7| 9.4-7| 5.1-8| −2.7-4</td><td>1:18:04</td></tr>
<tr><td>tai30a | 1393 | 900;</td><td>39| 39|1614</td><td>1.70671520 6 | 1.70679434 6</td><td>2.8-11| 7.4-7| 3.4-7| 9.9-7| 6.5-7| 2.2-7| −2.3-5</td><td>29:11</td></tr>
<tr><td>tai30b | 1393 | 900;</td><td>236|342|16584</td><td>5.98852630 8 | 5.99068570 8</td><td>9.0-7| 9.9-7| 6.9-14| 8.5-7| 6.1-15| 2.3-9| −1.8-4</td><td>2:52:00</td></tr>
<tr><td>tai35a | 1888 | 1225;</td><td>38| 38|3467</td><td>2.21649346 6 | 2.21657164 6</td><td>4.8-11| 6.6-7| 2.9-7| 9.9-7| 5.1-7| 2.4-7| −1.8-5</td><td>1:56:18</td></tr>
<tr><td>tai35b | 1888 | 1225;</td><td>142|214|10915</td><td>2.69644456 8 | 2.69710521 8</td><td>2.6-10| 9.9-7| 1.7-7| 9.8-7| 7.7-7| 6.7-8| −1.2-4</td><td>8:01:01</td></tr>
<tr><td>tai40a | 2458 | 1600;</td><td>33| 33|3395</td><td>2.84310602 6 | 2.84321095 6</td><td>7.6-11| 3.5-7| 2.8-7| 9.9-7| 6.0-7| 2.0-7| −1.8-5</td><td>3:56:34</td></tr>
<tr><td>tai40b | 2458 | 1600;</td><td>101|146|7124</td><td>6.09005347 8 | 6.09143489 8</td><td>5.6-10| 9.9-7| 2.5-7| 9.9-7| 9.9-7| 2.4-8| −1.1-4</td><td>10:55:44</td></tr>
<tr><td>tho30 | 1393 | 900;</td><td>44| 74|2925</td><td>1.43549788 5 | 1.43563445 5</td><td>6.3-11| 9.9-7| 1.8-7| 9.9-7| 5.1-7| 6.9-8| −4.8-5</td><td>1:03:01</td></tr>
</table>

Table 2 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>tho40 | 2458 | 1600;</td><td>24| 51|3998</td><td>2.26485088 5 | 2.26503953 5</td><td>2.0-10| 9.9-7| 2.0-7| 9.9-7| 5.2-7| 2.7-8| −4.2-5</td><td>5:08:15</td></tr>
<tr><td>be250.1 | 251 | 251;</td><td>122|123|2800</td><td>−2.51194635 4 |-2.51194398 4</td><td>1.1-7| 9.9-7| 1.3-7| 0.2-16| 1.8-8| 4.6-16| −4.7-7</td><td>1:13</td></tr>
<tr><td>be250.2 | 251 | 251;</td><td>121|121|2842</td><td>−2.36814919 4 |-2.36814545 4</td><td>1.2-12| 9.9-7| 1.1-8| 2.0-8| 1.5-8| 2.3-8| −7.9-7</td><td>1:12</td></tr>
<tr><td>be250.3 | 251 | 251;</td><td>84| 89|2200</td><td>−2.40000031 4 |-2.39999662 4</td><td>1.3-7| 9.9-7| 1.2-7| 0.1-16| 1.2-8| 1.4-16| −7.7-7</td><td>59</td></tr>
<tr><td>be250.4 | 251 | 251;</td><td>208|209|3850</td><td>−2.57203185 4 |-2.57202544 4</td><td>9.7-9| 9.9-7| 5.5-8| 0.0-16| 2.0-8| 0.2-16| −1.2-6</td><td>1:42</td></tr>
<tr><td>be250.5 | 251 | 251;</td><td>115|127|2791</td><td>−2.23747084 4 |-2.23746795 4</td><td>4.1-12| 9.9-7| 3.1-9| 3.4-8| 8.2-9| 2.9-7| −6.5-7</td><td>1:15</td></tr>
<tr><td>bqp500-1 | 501 | 501;</td><td>138|171|2499</td><td>−1.25964032 5 |-1.25964547 5</td><td>1.6-11| 9.9-7| 6.5-9| 6.8-9| 9.1-7| 1.6-7| 2.0-6</td><td>5:20</td></tr>
<tr><td>bqp500-2 | 501 | 501;</td><td>142|194|2390</td><td>−1.36011042 5 |-1.36011154 5</td><td>6.8-8| 9.9-7| 8.2-8| 0.1-16| 4.8-9| 0.3-16| 4.1-7</td><td>5:29</td></tr>
<tr><td>bqp500-3 | 501 | 501;</td><td>135|180|2390</td><td>−1.38453338 5 |-1.38453549 5</td><td>2.0-8| 9.7-7| 3.8-7| 0.5-16| 6.6-8| 2.8-16| 7.6-7</td><td>6:31</td></tr>
<tr><td>bqp500-4 | 501 | 501;</td><td>128|174|2390</td><td>−1.39328333 5 |-1.39328503 5</td><td>2.3-7| 9.9-7| 7.1-8| 0.2-16| 2.1-9| 6.0-16| 6.1-7</td><td>6:08</td></tr>
<tr><td>bqp500-5 | 501 | 501;</td><td>169|206|2910</td><td>−1.34092095 5 |-1.34092382 5</td><td>4.5-8| 9.9-7| 4.8-8| 0.0-16| 1.0-8| 0.2-16| 1.1-6</td><td>7:25</td></tr>
<tr><td>gka1f | 501 | 501;</td><td>166|203|2780</td><td>−6.55590598 4 |-6.55591369 4</td><td>1.3-8| 9.8-7| 1.6-7| 0.3-16| 1.3-8| 6.3-16| 5.9-7</td><td>6:32</td></tr>
<tr><td>gka2f | 501 | 501;</td><td>205|242|3541</td><td>−1.07931739 5 |-1.07932064 5</td><td>1.0-11| 9.9-7| 6.0-9| 9.1-9| 1.6-7| 8.1-8| 1.5-6</td><td>7:54</td></tr>
<tr><td>gka3f | 501 | 501;</td><td>174|216|2954</td><td>−1.50150987 5 |-1.50151193 5</td><td>6.4-12| 9.9-7| 1.5-8| 3.4-8| 5.2-8| 6.6-7| 6.8-7</td><td>6:51</td></tr>
<tr><td>gka4f | 501 | 501;</td><td>183|222|3101</td><td>−1.87087878 5 |-1.87087908 5</td><td>4.2-12| 9.9-7| 4.2-9| 2.4-8| 2.4-8| 2.0-7| 8.2-8</td><td>7:10</td></tr>
<tr><td>gka5f | 501 | 501;</td><td>142|187|2520</td><td>−2.06914264 5 |-2.06914258 5</td><td>1.8-7| 9.9-7| 7.1-8| 0.2-16| 1.3-8| 1.9-16| −1.5-8</td><td>5:53</td></tr>
<tr><td>soybean-large.2 | 308 | 307;</td><td>2| 2|1171</td><td>5.46342122 3 | 5.46342235 3</td><td>8.1-13| 9.2-7| 8.0-9| 9.9-7| 1.8-8| 6.6-7| −1.0-7</td><td>29</td></tr>
<tr><td>soybean-large.3 | 308 | 307;</td><td>2| 2|934</td><td>4.57580592 3 | 4.57580844 3</td><td>4.6-13| 7.2-7| 1.7-8| 2.7-7| 4.5-8| 9.3-8| −2.8-7</td><td>25</td></tr>
<tr><td>soybean-large.4 | 308 | 307;</td><td>52| 52|1506</td><td>4.04637305 3 | 4.04637422 3</td><td>1.0-13| 7.7-7| 2.8-7| 8.7-7| 2.7-8| 3.0-7| −1.4-7</td><td>52</td></tr>
<tr><td>soybean-large.5 | 308 | 307;</td><td>2| 2|814</td><td>3.63158072 3 | 3.63158133 3</td><td>2.6-13| 9.8-7| 0 | 9.6-7| 1.7-8| 2.0-7| −8.4-8</td><td>22</td></tr>
<tr><td>soybean-large.6 | 308 | 307;</td><td>0| 0|413</td><td>3.26767677 3 | 3.26767798 3</td><td>4.1-12| 9.4-7| 1.3-7| 5.7-7| 4.3-7| 1.1-7| −1.9-7</td><td>12</td></tr>
<tr><td>spambase-large.2 | 1501 | 1500;</td><td>0| 0|535</td><td>4.71138593 8 | 4.71150439 8</td><td>9.9-7| 9.9-7| 1.6-15| 3.0-7| 4.3-16| 2.0-7| −1.3-5</td><td>11:07</td></tr>
<tr><td>spambase-large.3 | 1501 | 1500;</td><td>8| 8|1844</td><td>2.36009657 8 | 2.36013239 8</td><td>2.5-10| 8.9-7| 2.3-7| 9.9-7| 6.0-7| 5.3-8| −7.6-6</td><td>1:40:31</td></tr>
<tr><td>spambase-large.4 | 1501 | 1500;</td><td>8| 8|4519</td><td>1.39698995 8 | 1.39699718 8</td><td>8.7-10| 9.8-7| 0 | 9.9-7| 6.1-9| 6.7-8| −2.6-6</td><td>2:49:39</td></tr>
</table>

Table 2 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>spambase-large.5 | 1501 | 1500;</td><td>8| 8|9184</td><td>1.02748129 8 | 1.02754393 8</td><td>3.8-13| 9.7-7| 2.6-8| 5.7-7| 2.5-8| 9.6-7| −3.0-5</td><td>4:49:37</td></tr>
<tr><td>spambase-large.6 | 1501 | 1500;</td><td>8| 8|2798</td><td>7.27756732 7 | 7.27685611 7</td><td>8.0-12| 9.9-7| 0 | 9.1-7| 4.5-7| 8.6-7| 4.9-5</td><td>2:07:59</td></tr>
<tr><td>abalone-large.2 | 1001 | 1000;</td><td>0| 0|576</td><td>5.52269325 3 | 5.52256503 3</td><td>9.9-7| 5.2-7| 1.4-15| 2.2-7| 1.8-15| 1.0-7| 1.2-5</td><td>5:01</td></tr>
<tr><td>abalone-large.3 | 1001 | 1000;</td><td>21| 21|762</td><td>2.81040989 3 | 2.81042183 3</td><td>2.1-13| 9.2-7| 7.6-7| 6.4-7| 3.9-8| 2.2-7| −2.1-6</td><td>7:29</td></tr>
<tr><td>abalone-large.4 | 1001 | 1000;</td><td>0| 0|545</td><td>1.72764378 3 | 1.72763706 3</td><td>2.7-11| 4.9-7| 0 | 2.0-7| 9.9-7| 7.9-8| 1.9-6</td><td>6:43</td></tr>
<tr><td>abalone-large.5 | 1001 | 1000;</td><td>38| 38|797</td><td>1.21466288 3 | 1.21471600 3</td><td>3.3-12| 9.5-7| 1.3-7| 4.6-7| 7.6-7| 2.6-9| −2.2-5</td><td>11:45</td></tr>
<tr><td>abalone-large.6 | 1001 | 1000;</td><td>8| 8|781</td><td>9.17362617 2 | 9.17389149 2</td><td>6.7-11| 9.9-7| 0 | 6.6-7| 5.2-7| 2.3-7| −1.4-5</td><td>9:12</td></tr>
<tr><td>segment-large.2 | 1001 | 1000;</td><td>8| 8|1191</td><td>1.47176055 7 | 1.47174710 7</td><td>9.7-12| 9.4-7| 0 | 1.6-7| 9.9-7| 6.5-8| 4.6-6</td><td>9:16</td></tr>
<tr><td>segment-large.3 | 1001 | 1000;</td><td>0| 0|373</td><td>1.03929738 7 | 1.03929372 7</td><td>6.1-12| 9.9-7| 0 | 9.7-7| 3.9-7| 1.1-7| 1.8-6</td><td>2:43</td></tr>
<tr><td>segment-large.4 | 1001 | 1000;</td><td>2| 2|1879</td><td>8.16944543 6 | 8.16945493 6</td><td>7.0-12| 9.0-7| 1.3-9| 9.9-7| 3.7-9| 2.1-7| −5.8-7</td><td>13:52</td></tr>
<tr><td>segment-large.5 | 1001 | 1000;</td><td>8| 8|2449</td><td>6.98489394 6 | 6.98490266 6</td><td>1.2-11| 9.9-7| 2.3-9| 9.7-7| 6.8-9| 2.2-7| −6.2-7</td><td>19:06</td></tr>
<tr><td>segment-large.6 | 1001 | 1000;</td><td>8| 8|3158</td><td>6.09809592 6 | 6.09811370 6</td><td>2.5-11| 8.8-7| 0 | 9.9-7| 3.7-9| 2.6-7| −1.5-6</td><td>24:00</td></tr>
<tr><td>housing.2 | 507 | 506;</td><td>8| 8|3373</td><td>5.76086706 6 | 5.76093491 6</td><td>9.9-7| 9.9-7| 1.4-15| 8.7-8| 3.2-15| 3.8-8| −5.9-6</td><td>4:50</td></tr>
<tr><td>housing.3 | 507 | 506;</td><td>8| 8|1576</td><td>3.00980144 6 | 3.00979147 6</td><td>4.5-12| 8.6-7| 0 | 7.0-8| 9.7-7| 1.2-7| 1.7-6</td><td>3:20</td></tr>
<tr><td>housing.4 | 507 | 506;</td><td>8| 8|1645</td><td>1.79283384 6 | 1.79284813 6</td><td>7.5-12| 9.9-7| 2.8-8| 2.3-8| 8.3-8| 8.5-9| −4.0-6</td><td>2:50</td></tr>
<tr><td>housing.5 | 507 | 506;</td><td>8| 8|1918</td><td>1.38028143 6 | 1.38019123 6</td><td>7.4-12| 9.9-7| 0 | 7.5-8| 9.5-7| 1.6-7| 3.3-5</td><td>3:30</td></tr>
<tr><td>housing.6 | 507 | 506;</td><td>11| 11|533</td><td>1.11181933 6 | 1.11182191 6</td><td>6.5-13| 9.9-7| 4.4-7| 9.6-7| 8.2-7| 1.8-7| −1.2-6</td><td>1:06</td></tr>
</table>

Table 3 Performance of SDPNAL+ on θ and R1TA problems ($\varepsilon = 10^{-6}$)

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>theta10 | 12470 | 500;</td><td>11| 11|200</td><td>8.38059601 1 | 8.38059488 1</td><td>4.4-8| 7.6-7| 6.5-16| 0 | 4.0-16| 0 | 6.7-8</td><td>32</td></tr>
<tr><td>theta102 | 37467 | 500;</td><td>11| 11| 84</td><td>3.83905392 1 | 3.83905464 1</td><td>8.1-8| 6.8-7| 3.4-16| 0 | 0.4-16| 0 | −9.3-8</td><td>21</td></tr>
<tr><td>theta103 | 62516 | 500;</td><td>20| 20| 64</td><td>2.25285667 1 | 2.25285686 1</td><td>2.8-7| 8.1-7| 1.8-16| 0 | 7.6-16| 0 | −4.1-8</td><td>38</td></tr>
<tr><td>theta104 | 87245 | 500;</td><td>43| 47| 63</td><td>1.33361259 1 | 1.33361411 1</td><td>3.9-7| 4.4-7| 1.3-16| 0 | 1.5-16| 0 | −5.5-7</td><td>53</td></tr>
<tr><td>theta12 | 17979 | 600;</td><td>13| 13|200</td><td>9.28016817 1 | 9.28016713 1</td><td>2.3-8| 3.7-7| 0.9-15| 0 | 1.3-16| 0 | 5.6-8</td><td>51</td></tr>
<tr><td>theta123 | 90020 | 600;</td><td>12| 12| 70</td><td>2.46686527 1 | 2.46686518 1</td><td>6.0-8| 7.1-7| 2.1-16| 0 | 1.1-16| 0 | 1.9-8</td><td>36</td></tr>
<tr><td>hamming-9-8 | 2305 | 512;</td><td>12| 12|200</td><td>2.24000000 2 | 2.24000041 2</td><td>1.1-9| 2.5-7| 7.1-16| 0 | 4.4-16| 0 | −9.2-8</td><td>21</td></tr>
<tr><td>hamming-10-2 | 23041 | 1024;</td><td>10| 10|200</td><td>1.02399926 2 | 1.02400047 2</td><td>1.2-8| 4.8-7| 1.3-16| 0 | 0.8-16| 0 | −5.9-7</td><td>2:54</td></tr>
<tr><td>hamming-9-5-6 | 53761 | 512;</td><td>6| 6|200</td><td>8.53333333 1 | 8.53333618 1</td><td>3.4-14| 6.2-7| 5.9-16| 0 | 0.2-16| 0 | −1.7-7</td><td>20</td></tr>
<tr><td>G43 | 9991 | 1000;</td><td>27| 27|200</td><td>2.80625087 2 | 2.80624586 2</td><td>6.8-7| 9.8-7| 1.6-15| 0 | 1.6-14| 0 | 8.9-7</td><td>3:32</td></tr>
<tr><td>G44 | 9991 | 1000;</td><td>30| 30|200</td><td>2.80583223 2 | 2.80583220 2</td><td>1.1-7| 6.1-7| 2.5-15| 0 | 0.8-16| 0 | 5.9-9</td><td>3:48</td></tr>
<tr><td>G45 | 9991 | 1000;</td><td>26| 26|200</td><td>2.80185148 2 | 2.80185127 2</td><td>1.3-7| 6.4-7| 3.2-15| 0 | 2.4-15| 0 | 3.8-8</td><td>3:31</td></tr>
<tr><td>G46 | 9991 | 1000;</td><td>28| 30|200</td><td>2.79837009 2 | 2.79836974 2</td><td>1.9-7| 7.9-7| 5.7-15| 0 | 5.4-15| 0 | 6.2-8</td><td>3:49</td></tr>
<tr><td>G47 | 9991 | 1000;</td><td>30| 31|200</td><td>2.81894037 2 | 2.81893954 2</td><td>1.8-7| 3.7-7| 1.4-14| 0 | 1.4-14| 0 | 1.5-7</td><td>3:52</td></tr>
<tr><td>G51 | 5910 | 1000;</td><td>148|584|200</td><td>3.48999920 2 | 3.48999975 2</td><td>7.3-7| 5.9-7| 9.2-14| 0 | 4.6-14| 0 | −7.9-8</td><td>41:06</td></tr>
<tr><td>G52 | 5917 | 1000;</td><td>458|1619|200</td><td>3.48387855 2 | 3.48386488 2</td><td>7.6-7| 9.3-7| 3.0-15| 0 | 2.4-14| 0 | 2.0-6</td><td>3:53:19</td></tr>
<tr><td>G53 | 5915 | 1000;</td><td>425|1183|200</td><td>3.48348615 2 | 3.48347655 2</td><td>4.7-7| 9.9-7| 8.9-14| 0 | 3.2-15| 0 | 1.4-6</td><td>2:16:35</td></tr>
<tr><td>G54 | 5917 | 1000;</td><td>123|462|200</td><td>3.41000018 2 | 3.40999990 2</td><td>2.5-7| 9.3-7| 9.3-15| 0 | 5.7-16| 0 | 4.2-8</td><td>23:17</td></tr>
<tr><td>1dc.1024 | 24064 | 1024;</td><td>48| 74|200</td><td>9.59856502 1 | 9.59849891 1</td><td>9.1-7| 5.9-7| 5.5-14| 0 | 3.8-14| 0 | 3.4-6</td><td>14:32</td></tr>
<tr><td>1et.1024 | 9601 | 1024;</td><td>64|129|200</td><td>1.84226960 2 | 1.84226147 2</td><td>5.9-7| 3.3-7| 5.3-15| 0 | 6.1-14| 0 | 2.2-6</td><td>35:33</td></tr>
<tr><td>1tc.1024 | 7937 | 1024;</td><td>156|417|200</td><td>2.06304654 2 | 2.06304284 2</td><td>7.5-7| 6.2-7| 3.1-14| 0 | 7.5-16| 0 | 8.9-7</td><td>1:22:24</td></tr>
<tr><td>1zc.1024 | 16641 | 1024;</td><td>16| 16|200</td><td>1.28666672 2 | 1.28666665 2</td><td>1.8-8| 8.4-7| 0.8-15| 0 | 3.8-15| 0 | 2.7-8</td><td>4:56</td></tr>
<tr><td>2dc.1024 | 169163 | 1024;</td><td>148|376|200</td><td>1.86381938 1 | 1.86378972 1</td><td>8.2-7| 9.2-7| 1.1-14| 0 | 4.0-15| 0 | 7.8-6</td><td>2:21:20</td></tr>
<tr><td>1dc.2048 | 58368 | 2048;</td><td>62|112|200</td><td>1.74731330 2 | 1.74729390 2</td><td>9.7-7| 6.2-7| 5.3-14| 0 | 1.6-14| 0 | 5.5-6</td><td>1:55:31</td></tr>
</table>

Table 3 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>1et.2048 | 22529 | 2048;</td><td>228|658|200</td><td>3.42031726 2 | 3.42029072 2</td><td>8.7-7| 8.0-7| 1.1-14| 0 | 4.8-15| 0 | 3.9-6</td><td>5:29:46</td></tr>
<tr><td>1tc.2048 | 18945 | 2048;</td><td>509|1725|200</td><td>3.74644012 2 | 3.74643271 2</td><td>4.7-7| 8.2-7| 1.6-14| 0 | 9.6-15| 0 | 9.9-7</td><td>22:10:45</td></tr>
<tr><td>2dc.2048 | 504452 | 2048;</td><td>167|385|200</td><td>3.06739144 1 | 3.06728102 1</td><td>9.8-7| 8.2-7| 3.2-15| 0 | 2.7-15| 0 | 1.8-5</td><td>14:22:06</td></tr>
<tr><td>nonsym(8,4) | 46655 | 512;</td><td>13| 13|200</td><td>5.74083101 0 | 5.74083615 0</td><td>1.5-7| 2.4-8| 0 | 0 | 1.0-16| 0 | −4.1-7</td><td>29</td></tr>
<tr><td>nonsym(9,4) | 91124 | 729;</td><td>25| 33|200</td><td>1.06613340 0 | 1.06611027 0</td><td>8.5-8| 4.2-7| 0 | 0 | 3.0-16| 0 | 7.4-6</td><td>2:00</td></tr>
<tr><td>nonsym(10,4) | 166374 | 1000;</td><td>17| 21|200</td><td>1.69471772 0 | 1.69472878 0</td><td>7.9-7| 1.6-7| 0.8-16| 0 | 0.5-16| 0 | −2.5-6</td><td>3:37</td></tr>
<tr><td>nonsym(11,4) | 287495 | 1331;</td><td>19| 22|200</td><td>2.91348466 0 | 2.91343357 0</td><td>7.5-8| 2.5-7| 0.2-16| 0 | 0.6-16| 0 | 7.5-6</td><td>7:14</td></tr>
<tr><td>nonsym(5,5) | 50624 | 625;</td><td>30| 31|200</td><td>3.08257539 0 | 3.08254910 0</td><td>2.4-7| 1.7-7| 1.8-16| 0 | 1.3-16| 0 | 3.7-6</td><td>1:18</td></tr>
<tr><td>nonsym(6,5) | 194480 | 1296;</td><td>26| 28|200</td><td>3.09572024 0 | 3.09557416 0</td><td>6.4-7| 6.6-7| 0.8-16| 0 | 0.1-16| 0 | 2.0-5</td><td>6:39</td></tr>
<tr><td>sym_rd(3,35) | 82250 | 666;</td><td>43| 46| 72</td><td>1.82999132 0 | 1.83002862 0</td><td>5.4-7| 3.3-7| 0 | 0 | 0.8-16| 0 | −8.0-6</td><td>1:27</td></tr>
<tr><td>sym_rd(3,40) | 135750 | 861;</td><td>37| 39| 82</td><td>1.99315221 0 | 1.99323089 0</td><td>1.7-7| 5.7-7| 0.3-16| 0 | 1.0-16| 0 | −1.6-5</td><td>2:18</td></tr>
<tr><td>sym_rd(3,45) | 211875 | 1081;</td><td>31| 31| 87</td><td>2.14076548 0 | 2.14073540 0</td><td>5.5-7| 2.2-7| 0 | 0 | 1.1-16| 0 | 5.7-6</td><td>4:31</td></tr>
<tr><td>sym_rd(3,50) | 316250 | 1326;</td><td>37| 37| 87</td><td>2.06951100 0 | 2.06938546 0</td><td>5.1-7| 7.6-7| 0 | 0 | 0.1-16| 0 | 2.4-5</td><td>8:24</td></tr>
<tr><td>sym_rd(4,35) | 73814 | 630;</td><td>46| 51| 77</td><td>1.09833279 1 | 1.09831864 1</td><td>6.3-8| 2.2-7| 0 | 0 | 0.7-16| 0 | 6.2-6</td><td>1:45</td></tr>
<tr><td>sym_rd(4,40) | 123409 | 820;</td><td>60| 76| 86</td><td>1.15471518 1 | 1.15473381 1</td><td>5.4-8| 6.6-7| 1.6-13| 0 | 3.2-16| 0 | −7.7-6</td><td>4:56</td></tr>
<tr><td>sym_rd(4,45) | 194579 | 1035;</td><td>45| 62| 90</td><td>1.18424653 1 | 1.18425819 1</td><td>5.7-8| 3.5-7| 1.0-13| 0 | 1.3-16| 0 | −4.7-6</td><td>7:44</td></tr>
<tr><td>sym_rd(4,50) | 292824 | 1275;</td><td>45| 62| 91</td><td>1.30418148 1 | 1.30421731 1</td><td>6.8-8| 7.9-7| 1.0-15| 0 | 1.4-16| 0 | −1.3-5</td><td>12:44</td></tr>
<tr><td>sym_rd(5,15) | 54263 | 816;</td><td>41| 43|111</td><td>3.49345457 0 | 3.49338911 0</td><td>5.3-8| 2.4-7| 0 | 0 | 0.3-16| 0 | 8.2-6</td><td>2:53</td></tr>
<tr><td>sym_rd(5,20) | 230229 | 1771;</td><td>29| 35|139</td><td>4.17928037 0 | 4.17942269 0</td><td>3.1-7| 2.9-7| 0 | 0 | 0.8-16| 0 | −1.5-5</td><td>27:28</td></tr>
<tr><td>sym_rd(6,15) | 38759 | 680;</td><td>32| 35|137</td><td>2.70987911 1 | 2.70973348 1</td><td>4.7-7| 6.4-7| 0 | 0 | 1.2-16| 0 | 2.6-5</td><td>1:33</td></tr>
<tr><td>sym_rd(6,20) | 177099 | 1540;</td><td>26| 37|185</td><td>3.15086704 1 | 3.15084788 1</td><td>6.4-7| 4.2-7| 5.3-16| 0 | 1.2-16| 0 | 3.0-6</td><td>22:48</td></tr>
<tr><td>nsym_rd([20,25,25]) | 68249 | 500;</td><td>47| 51| 66</td><td>2.78568871 0 | 2.78562272 0</td><td>2.4-7| 6.0-7| 0 | 0 | 2.0-16| 0 | 1.0-5</td><td>58</td></tr>
<tr><td>nsym_rd([25,20,25]) | 68249 | 500;</td><td>49| 50| 77</td><td>2.77557008 0 | 2.77563681 0</td><td>8.4-8| 7.0-7| 0.3-16| 0 | 3.8-16| 0 | −1.0-5</td><td>58</td></tr>
<tr><td>nsym_rd([25,25,20]) | 68249 | 500;</td><td>14| 14|129</td><td>2.87657149 0 | 2.87658217 0</td><td>8.8-8| 9.0-8| 0 | 0 | 1.4-16| 0 | −1.6-6</td><td>34</td></tr>
</table>

Table 3 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>it|itsub|itA</th><th>$pobj$ | $dobj$</th><th>η_P | η_D | $\eta_{\mathcal{K}_1}$ | $\eta_{\mathcal{K}_2}$ | η_{C1} | η_{C2} | η_g</th><th>Time</th></tr>
<tr><td>nsym_rd([25,25,25]) | 105624 | 625;</td><td>56| 64| 95</td><td>2.83000532 0 | 2.83011896 0</td><td>1.5-7| 9.0-7| 0 | 0 | 1.7-16| 0 | −1.7-5</td><td>1:33</td></tr>
<tr><td>nsym_rd([30,30,30]) | 216224 | 900;</td><td>30| 33|122</td><td>3.03772558 0 | 3.03770676 0</td><td>6.2-7| 4.6-7| 0.3-16| 0 | 0.0-16| 0 | 2.7-6</td><td>3:52</td></tr>
<tr><td>nsym_rd([35,35,35]) | 396899 | 1225;</td><td>45| 49|141</td><td>3.07047975 0 | 3.07050501 0</td><td>1.4-7| 2.0-7| 0 | 0 | 1.4-16| 0 | −3.5-6</td><td>9:14</td></tr>
<tr><td>nsym_rd([40,40,40]) | 672399 | 1600;</td><td>33| 35| 93</td><td>3.87873078 0 | 3.87863899 0</td><td>2.1-7| 3.3-7| 0.2-16| 0 | 0.1-16| 0 | 1.0-5</td><td>14:23</td></tr>
<tr><td>nsym_rd([8,8,8,8]) | 46655 | 512;</td><td>11| 11|200</td><td>2.83768958 0 | 2.83773386 0</td><td>1.0-7| 3.3-7| 0 | 0 | 1.0-16| 0 | −6.6-6</td><td>28</td></tr>
<tr><td>nsym_rd([9,9,9,9]) | 91124 | 729;</td><td>14| 14|200</td><td>3.10895856 0 | 3.10890841 0</td><td>3.1-7| 2.6-7| 2.6-16| 0 | 2.0-16| 0 | 6.9-6</td><td>1:07</td></tr>
<tr><td>nonsym(12,4) | 474551 | 1728;</td><td>5| 17|200</td><td>5.92161950 0 | 5.92162092 0</td><td>2.8-8| 4.5-9| 0.1-16| 0 | 0.2-16| 0 | −1.1-7</td><td>16:55</td></tr>
<tr><td>nonsym(13,4) | 753570 | 2197;</td><td>15| 55|200</td><td>7.27450656 0 | 7.27450942 0</td><td>5.2-7| 5.6-9| 1.0-16| 0 | 0.8-16| 0 | −1.8-7</td><td>1:51:34</td></tr>
<tr><td>nonsym(7,5) | 614655 | 2401;</td><td>32| 43|200</td><td>5.10582689 0 | 5.10572890 0</td><td>2.4-7| 2.0-7| 0 | 0 | 2.7-16| 0 | 8.7-6</td><td>53:29</td></tr>
<tr><td>nonsym(8,5) | 1679615 | 4096;</td><td>14| 22|200</td><td>5.77855140 0 | 5.77862086 0</td><td>5.2-7| 1.0-7| 0 | 0 | 4.8-16| 0 | −5.5-6</td><td>2:46:20</td></tr>
<tr><td>nonsym(18,4) | 5000210 | 5832;</td><td>13| 55|200</td><td>1.53963123 1 | 1.53954727 1</td><td>5.8-7| 3.7-7| 0 | 0 | 1.6-16| 0 | 2.6-5</td><td>8:50:14</td></tr>
<tr><td>nonsym(20,4) | 9260999 | 8000;</td><td>7| 17|200</td><td>1.77231047 1 | 1.77233375 1</td><td>5.2-8| 7.4-8| 0 | 0 | 2.2-15| 0 | −6.4-6</td><td>8:26:40</td></tr>
<tr><td>nonsym(21,4) | 12326390 | 9261;</td><td>7| 21|200</td><td>2.03462783 1 | 2.03463278 1</td><td>5.7-8| 1.3-8| 2.9-15| 0 | 2.9-15| 0 | −1.2-6</td><td>14:22:05</td></tr>
</table>

Table 4 Performance of SDPNAL+ (a), ADMM+ (b), SDPAD (c) and 2EBD (d) on θ_+, FAP, QAP, BIQ and RCP problems ($\varepsilon = 10^{-6}$)

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c|d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>theta10 | 12470 | 500;</td><td>0, 0;354 | 354 | 351 | 490</td><td>8.5-7 | 8.5-7 | 9.9-7 | 9.9-7</td><td>−2.5-6 | −2.5-6 | −7.8-7 | 1.5-6</td><td>46 | 45 | 42 | 1:11</td></tr>
<tr><td>theta102 | 37467 | 500;</td><td>0, 0;157 | 157 | 130 | 355</td><td>9.5-7 | 9.5-7 | 9.0-7 | 9.9-7</td><td>−6.0-7 | −6.0-7 | 2.1-6 | 2.2-6</td><td>23 | 23 | 23 | 54</td></tr>
<tr><td>theta103 | 62516 | 500;</td><td>0, 0;144 | 144 | 108 | 323</td><td>9.2-7 | 9.2-7 | 9.8-7 | 9.8-7</td><td>−3.0-8 | −3.0-8 | 5.9-7 | 2.6-6</td><td>22 | 22 | 21 | 49</td></tr>
<tr><td>theta104 | 87245 | 500;</td><td>0, 0;169 | 169 | 123 | 338</td><td>9.3-7 | 9.3-7 | 9.8-7 | 9.9-7</td><td>−9.2-8 | −9.2-8 | 1.1-6 | 3.2-6</td><td>24 | 24 | 20 | 51</td></tr>
<tr><td>theta12 | 17979 | 600;</td><td>0, 0;362 | 362 | 366 | 494</td><td>9.0-7 | 9.0-7 | 8.8-7 | 9.2-7</td><td>−2.2-6 | −2.2-6 | 1.0-6 | 1.3-6</td><td>1:15 | 1:14 | 1:11 | 1:52</td></tr>
<tr><td>theta123 | 90020 | 600;</td><td>0, 0;156 | 156 | 107 | 345</td><td>9.3-7 | 9.3-7 | 9.9-7 | 9.9-7</td><td>−6.0-8 | −6.0-8 | 5.1-7 | 2.6-6</td><td>34 | 35 | 33 | 1:26</td></tr>
<tr><td>hamming-9-8 | 2305 | 512;</td><td>11,11;500 | 2413 | 3100 | 938</td><td>9.5-7 | 9.6-7 | 9.6-7 | 9.0-7</td><td>−4.4-8 | −1.2-5 | −6.9-7 | 5.6-6</td><td>44 | 3:07 | 4:20 | 1:36</td></tr>
<tr><td>hamming-10-2 | 23041 | 1024;</td><td>0, 0;657 | 657 | 651 | 902</td><td>8.7-7 | 8.7-7 | 9.4-7 | 8.8-7</td><td>7.6-6 | 7.6-6 | −2.6-6 | 3.4-5</td><td>3:09 | 3:05 | 5:17 | 3:47</td></tr>
<tr><td>hamming-9-5-6 | 53761 | 512;</td><td>0, 0;461 | 461 | 507 | 563</td><td>9.5-7 | 9.5-7 | 9.5-7 | 8.9-7</td><td>−1.2-5 | −1.2-5 | −1.9-6 | 8.0-6</td><td>45 | 45 | 54 | 58</td></tr>
<tr><td>G43 | 9991 | 1000;</td><td>21,21;973 | 1154 | 1147 | 934</td><td>8.9-7 | 9.8-7 | 9.4-7 | 9.9-7</td><td>4.1-6 | −3.1-6 | 1.7-6 | 2.0-6</td><td>12:32 | 13:04 | 10:20 | 13:00</td></tr>
<tr><td>G44 | 9991 | 1000;</td><td>21,21;942 | 1151 | 1144 | 968</td><td>9.9-7 | 9.3-7 | 9.9-7 | 9.9-7</td><td>−5.1-6 | −2.9-6 | 1.6-6 | 1.6-6</td><td>12:00 | 12:13 | 10:11 | 13:15</td></tr>
<tr><td>G45 | 9991 | 1000;</td><td>21,21;888 | 1175 | 1185 | 966</td><td>9.8-7 | 9.5-7 | 9.4-7 | 9.9-7</td><td>−6.5-6 | 2.9-6 | −1.0-6 | 1.6-6</td><td>11:43 | 13:24 | 10:36 | 13:28</td></tr>
<tr><td>G46 | 9991 | 1000;</td><td>21,21;887 | 1199 | 1180 | 943</td><td>9.7-7 | 9.9-7 | 9.8-7 | 9.9-7</td><td>−1.1-5 | −3.2-6 | −1.0-6 | 1.4-6</td><td>11:35 | 12:55 | 10:42 | 12:58</td></tr>
<tr><td>G47 | 9991 | 1000;</td><td>21,21;1042 | 1186 | 1137 | 992</td><td>9.4-7 | 9.5-7 | 9.5-7 | 9.9-7</td><td>−5.4-6 | 2.9-6 | −9.4-7 | 1.2-6</td><td>13:10 | 13:18 | 10:28 | 13:50</td></tr>
<tr><td>G51 | 5910 | 1000;</td><td>1, 2;5672 | 6207 | 10361 | 9586</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.0-6 | 3.7-7 | 2.6-7 | 5.6-7</td><td>1:15:12 | 1:21:52 | 2:11:03 | 2:31:30</td></tr>
<tr><td>G52 | 5917 | 1000;</td><td>5, 5;10840 | 11463 | 14163 | 12124</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>2.4-7 | 4.2-7 | 4.5-7 | 6.9-7</td><td>2:21:46 | 2:26:28 | 2:46:25 | 3:15:11</td></tr>
<tr><td>G53 | 5915 | 1000;</td><td>4, 4;13260 | 13289 | 23865 | 20623</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>2.9-6 | 2.6-6 | 2.9-6 | 4.2-6</td><td>2:48:21 | 2:49:53 | 4:48:56 | 5:49:06</td></tr>
<tr><td>G54 | 5917 | 1000;</td><td>8, 8;4278 | 3262 | 7542 | 5136</td><td>9.9-7 | 9.7-7 | 9.9-7 | 9.9-7</td><td>−7.9-7 | 3.1-6 | 4.6-7 | 1.3-6</td><td>51:18 | 38:42 | 1:26:47 | 1:17:01</td></tr>
<tr><td>1dc.1024 | 24064 | 1024;</td><td>0, 0;2620 | 2620 | 2681 | 3641</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.3-6 | 1.3-6 | 3.4-6 | 4.0-6</td><td>31:46 | 32:22 | 45:21 | 53:12</td></tr>
<tr><td>1et.1024 | 9601 | 1024;</td><td>0, 0;1144 | 1144 | 2563 | 2609</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.3-6 | 1.3-6 | 5.6-6 | 5.9-6</td><td>12:43 | 12:54 | 39:53 | 35:35</td></tr>
<tr><td>1tc.1024 | 7937 | 1024;</td><td>0, 0;2732 | 2732 | 6545 | 6675</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>4.5-6 | 4.5-6 | 4.5-6 | 4.2-6</td><td>31:34 | 32:08 | 1:48:31 | 1:40:06</td></tr>
<tr><td>1zc.1024 | 16641 | 1024;</td><td>0, 0;711 | 711 | 770 | 25000</td><td>7.7-7 | 7.7-7 | 9.9-7 | 3.1-5</td><td>5.4-6 | 5.4-6 | 2.0-6 | 7.9-4</td><td>7:12 | 7:19 | 12:18 | 7:48:20</td></tr>
<tr><td>2dc.1024 | 169163 | 1024;</td><td>0, 0;4135 | 4135 | 1896 | 1891</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.3-5 | 1.3-5 | 1.0-5 | 1.5-5</td><td>44:55 | 45:59 | 29:02 | 24:59</td></tr>
</table>

Table 4 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c|d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>1dc.2048 | 58368 | 2048;</td><td>0, 0;4153 | 4153 | 7277 | 8476</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>4.2-6 | 4.2-6 | 6.4-6 | 6.5-6</td><td>5:50:06 | 5:47:45 | 13:59:49 | 16:04:13</td></tr>
<tr><td>1et.2048 | 22529 | 2048;</td><td>0, 0;3039 | 3039 | 4422 | 4739</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.1-6 | 1.1-6 | 4.8-6 | 7.8-6</td><td>4:01:54 | 4:04:34 | 8:47:18 | 8:28:46</td></tr>
<tr><td>1tc.2048 | 18945 | 2048;</td><td>0, 0;2876 | 2876 | 7329 | 7482</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.5-6 | 1.5-6 | 5.5-6 | 5.6-6</td><td>3:50:43 | 3:50:16 | 13:29:15 | 13:50:32</td></tr>
<tr><td>2dc.2048 | 504452 | 2048;</td><td>0, 0;2997 | 2997 | 2147 | 1849</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>8.3-6 | 8.3-6 | 1.0-5 | 2.2-5</td><td>3:54:58 | 3:52:42 | 4:13:47 | 3:07:46</td></tr>
<tr><td>fap11 | 252 | 252;</td><td>5, 5;1180 | 1559 | 2585 | 2771</td><td>9.6-7 | 5.3-7 | 9.9-7 | 9.7-7</td><td>−6.8-5 | −1.9-5 | **−2.2-4** | **−1.1-4**</td><td>39 | 50 | 1:07 | 1:18</td></tr>
<tr><td>fap12 | 369 | 369;</td><td>15,15;1768 | 1830 | 3394 | 3325</td><td>9.9-7 | 8.4-7 | 9.9-7 | 9.9-7</td><td>−6.6-5 | −2.6-5 | **−2.2-4** | **−1.3-4**</td><td>1:56 | 1:55 | 3:32 | 3:08</td></tr>
<tr><td>fap25 | 2118 | 2118;</td><td>11,11;2268 | 5799 | 5495 | 4498</td><td>9.2-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−8.2-5 | −3.2-5 | **−1.1-4** | −7.1-5</td><td>3:58:21 | 10:55:33 | 13:26:47 | 8:11:50</td></tr>
<tr><td>fap36 | 4110 | 4110;</td><td>4, 4;2033 | 2824 | 4445 | 3500</td><td>9.5-7 | 9.9-7 | 9.9-7 | 9.8-7</td><td>−2.5-5 | −1.7-5 | −3.0-5 | −2.8-5</td><td>23:07:56 | 30:57:53 | 78:43:03 | 43:37:44</td></tr>
<tr><td>bur26a | 1051 | 676;</td><td>137,222;10228 | 25000 | 25000 | 25000</td><td>9.9-7 | 5.6-6 | 1.1-5 | 8.9-6</td><td>−1.8-5 | −6.3-5 | −7.7-5 | −8.2-5</td><td>1:48:05 | 2:05:11 | 2:07:44 | 2:38:24</td></tr>
<tr><td>bur26b | 1051 | 676;</td><td>100,208;8605 | 25000 | 25000 | 25000</td><td>9.9-7 | 6.8-6 | 1.1-5 | 9.3-6</td><td>−1.8-5 | −5.7-5 | -8.0-5 | −7.5-5</td><td>1:32:52 | 2:07:13 | 1:57:30 | 2:49:59</td></tr>
<tr><td>bur26c | 1051 | 676;</td><td>247,441;21498 | 25000 | 25000 | 25000</td><td>9.9-7 | 4.2-6 | 1.4-5 | 1.4-5</td><td>−2.0-5 | −4.5-5 | **−1.2-4** | **−1.8-4**</td><td>2:03:12 | 2:05:11 | 2:02:35 | 2:50:08</td></tr>
<tr><td>bur26d | 1051 | 676;</td><td>173,306;13287 | 25000 | 25000 | 25000</td><td>9.9-7 | 6.4-6 | 1.5-5 | 1.3-5</td><td>−1.3-5 | −8.4-5 | **−1.2-4** | **−1.4-4**</td><td>1:59:20 | 2:02:24 | 1:51:20 | 2:53:07</td></tr>
<tr><td>bur26e | 1051 | 676;</td><td>129,361;14705 | 25000 | 25000 | 25000</td><td>9.4-7 | 3.1-6 | 6.4-6 | 1.4-5</td><td>−1.1-5 | −2.8-5 | −3.6-5 | **−1.9-4**</td><td>1:18:35 | 2:03:18 | 2:28:06 | 2:46:03</td></tr>
<tr><td>bur26f | 1051 | 676;</td><td>107,248;11272 | 20887 | 25000 | 25000</td><td>9.9-7 | 9.9-7 | 8.1-6 | 1.2-5</td><td>−1.0-5 | −1.0-5 | −4.8-5 | −7.5-5</td><td>1:45:13 | 1:45:08 | 2:09:11 | 2:44:28</td></tr>
<tr><td>bur26g | 1051 | 676;</td><td>250,392;10817 | 17910 | 25000 | 25000</td><td>9.9-7 | 8.6-7 | 1.6-6 | 7.8-6</td><td>−2.4-5 | −6.3-6 | −4.0-5 | −6.9-5</td><td>1:32:44 | 1:29:22 | 1:57:13 | 2:46:34</td></tr>
<tr><td>bur26h | 1051 | 676;</td><td>146,360;10658 | 23208 | 25000 | 25000</td><td>9.9-7 | 9.4-7 | 1.4-6 | 2.3-5</td><td>1.9-5 | −1.4-6 | −2.3-5 | **−1.7-4**</td><td>1:25:45 | 1:57:33 | 2:01:12 | 2:54:20</td></tr>
<tr><td>chr22a | 757 | 484;</td><td>104,250;6940 | 6457 | 22364 | 25000</td><td>3.5-7 | 8.8-7 | 9.9-7 | 2.5-5</td><td>6.6-6 | **3.9-4** | **−2.7-4** | **−2.6-3**</td><td>22:57 | 12:29 | 50:45 | 1:17:44</td></tr>
<tr><td>chr22b | 757 | 484;</td><td>89,189;5620 | 7211 | 25000 | 25000</td><td>3.1-7 | 9.7-7 | 1.5-5 | 2.1-5</td><td>1.7-6 | **3.4-4** | **−1.4-3** | **−1.8-3**</td><td>12:46 | 13:52 | 1:07:51 | 1:15:34</td></tr>
<tr><td>chr25a | 973 | 625;</td><td>53,200;5151 | 7127 | 25000 | 25000</td><td>8.6-7 | 8.6-7 | 3.7-5 | 3.4-5</td><td>**5.6-4** | **7.8-4** | **−9.5-3** | **−6.8-3**</td><td>21:04 | 26:03 | 2:10:29 | 2:23:01</td></tr>
<tr><td>esc32a | 1582 | 1024;</td><td>46,78;1664 | 4931 | 2247 | 9630</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.1-6 | −2.0-6 | −1.1-6 | −2.7-6</td><td>32:32 | 1:06:25 | 27:49 | 2:44:37</td></tr>
<tr><td>esc32b | 1582 | 1024;</td><td>52,100;2196 | 25000 | 25000 | 25000</td><td>9.8-7 | 3.7-6 | 2.4-6 | 8.3-6</td><td>−5.1-5 | **−2.4-4** | **−2.2-4** | **−4.0-4**</td><td>45:50 | 5:18:19 | 4:31:51 | 7:59:47</td></tr>
<tr><td>esc32c | 1582 | 1024;</td><td>46,139;3562 | 25000 | 25000 | 25000</td><td>9.7-7 | 5.3-6 | 5.1-6 | 1.6-5</td><td>−4.1-6 | −5.4-5 | −5.2-5 | −9.8-5</td><td>1:06:19 | 4:58:47 | 4:00:23 | 8:01:19</td></tr>
<tr><td>esc32d | 1582 | 1024;</td><td>0, 0;678 | 678 | 799 | 1412</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−9.4-6 | −9.4-6 | −1.7-5 | −2.2-7</td><td>9:40 | 9:39 | 8:01 | 25:40</td></tr>
<tr><td>esc32e | 1582 | 1024;</td><td>40,47;1248 | 1108 | 905 | 784</td><td>9.9-7 | 9.8-7 | 8.6-7 | 9.7-7</td><td>8.7-6 | −3.8-7 | −9.2-6 | −3.1-6</td><td>22:00 | 16:09 | 8:32 | 14:30</td></tr>
</table>

Table 4 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c|d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>esc32f | 1582 | 1024;</td><td>40,47;1248 | 1108 | 905 | 784</td><td>9.9-7 | 9.8-7 | 8.6-7 | 9.7-7</td><td>8.7-6 | −3.8-7 | −9.2-6 | −3.1-6</td><td>21:31 | 15:45 | 8:25 | 12:57</td></tr>
<tr><td>esc32g | 1582 | 1024;</td><td>0, 0;520 | 520 | 588 | 981</td><td>9.3-7 | 9.3-7 | 9.2-7 | 9.9-7</td><td>1.9-6 | 1.9-6 | −3.3-6 | 3.6-7</td><td>7:17 | 7:14 | 5:41 | 17:19</td></tr>
<tr><td>esc32h | 1582 | 1024;</td><td>97,236;4959 | 25000 | 25000 | 25000</td><td>9.9-7 | 9.4-6 | 1.2-5 | 2.8-5</td><td>−4.4-5 | **−4.1-4** | **−5.0-4** | **−7.4-4**</td><td>1:42:34 | 5:01:41 | 4:13:31 | 7:30:56</td></tr>
<tr><td>kra30a | 1393 | 900;</td><td>49,72;3208 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.3-5 | 1.4-5 | 1.7-6</td><td>−6.5-5 | **−4.2-4** | **−5.7-4** | −5.9-5</td><td>52:13 | 3:45:29 | 5:11:22 | 5:42:45</td></tr>
<tr><td>kra30b | 1393 | 900;</td><td>81,101;3080 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.1-5 | 1.1-5 | 1.8-5</td><td>−6.5-5 | **−3.7-4** | **−4.9-4** | **−6.1-4**</td><td>55:48 | 3:56:14 | 5:15:18 | 5:45:45</td></tr>
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<tr><td>lipa30b | 1393 | 900;</td><td>4, 9;820 | 2700 | 7516 | 25000</td><td>8.2-9 | 9.1-7 | 9.9-7 | **1.6-4**</td><td>−3.7-7 | 4.3-5 | 4.7-5 | **1.1-2**</td><td>11:08 | 16:33 | 52:22 | 5:31:56</td></tr>
<tr><td>lipa40a | 2458 | 1600;</td><td>153,546;3732 | 6483 | 18785 | 25000</td><td>5.5-7 | 8.8-7 | 9.8-7 | 9.5-6</td><td>1.2-6 | 4.2-5 | −4.1-5 | **−1.5-4**</td><td>3:01:05 | 3:32:47 | 19:15:19 | 26:56:27</td></tr>
<tr><td>lipa40b | 2458 | 1600;</td><td>5,18;991 | 4878 | 5970 | 25000</td><td>4.2-7 | 9.0-7 | 9.8-7 | **4.6-4**</td><td>1.9-7 | **1.1-4** | −6.4-5 | **−4.1-2**</td><td>1:02:59 | 2:20:09 | 4:11:06 | 23:33:11</td></tr>
<tr><td>nug24 | 898 | 576;</td><td>43,66;2359 | 25000 | 25000 | 25000</td><td>9.9-7 | 9.1-6 | 1.1-5 | 1.6-5</td><td>−2.8-5 | **−1.9-4** | **−2.3-4** | **−2.7-4**</td><td>14:12 | 1:17:49 | 1:40:05 | 1:57:03</td></tr>
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<tr><td>nug27 | 1132 | 729;</td><td>49,86;3300 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.0-5 | 1.3-5 | 1.7-5</td><td>−2.1-5 | **−2.0-4** | **−2.6-4** | **−2.8-4**</td><td>36:53 | 2:21:06 | 2:51:26 | 3:28:20</td></tr>
<tr><td>nug28 | 1216 | 784;</td><td>50,77;3190 | 25000 | 25000 | 25000</td><td>9.9-7 | 9.3-6 | 1.2-5 | 1.7-5</td><td>−2.0 − 5 | **−1.8-4** | **−2.2-4** | **−2.6-4**</td><td>40:26 | 2:47:04 | 3:27:54 | 4:02:11</td></tr>
<tr><td>nug30 | 1393 | 900;</td><td>44,68;2463 | 25000 | 25000 | 25000</td><td>9.9-7 | 8.7-6 | 1.1-5 | 1.7-5</td><td>−2.5-5 | **−1.6-4** | **−1.9-4** | **−2.2-4**</td><td>45:02 | 3:48:43 | 4:58:12 | 5:39:31</td></tr>
<tr><td>ste36a | 1996 | 1296;</td><td>122,189;7344 | 25000 | 25000 | 25000</td><td>9.9-7 | 9.7-6 | 1.3-5 | 1.6-5</td><td>−8.1-5 | **−5.8-4** | **−6.8-4** | **−6.7-4**</td><td>6:29:21 | 9:38:26 | 12:37:18 | 14:09:11</td></tr>
<tr><td>ste36b | 1996 | 1296;</td><td>173,242;11851 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.2-5 | 1.8-5 | 1.3-5</td><td>**−2.3-4** | **−1.5-3** | **−2.0-3** | **−2.1-3**</td><td>9:45:58 | 9:19:24 | 12:10:09 | 14:23:33</td></tr>
<tr><td>ste36c | 1996 | 1296;</td><td>143,202;10008 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.2-5 | 1.5-5 | 1.6-5</td><td>−8.6-5 | **−5.8-4** | **−7.3-4** | **−7.2-4**</td><td>8:06:50 | 9:26:42 | 12:22:19 | 14:23:52</td></tr>
<tr><td>tai25a | 973 | 625;</td><td>33,42;2630 | 2201 | 1845 | 25000</td><td>9.9-7 | 9.5-7 | 9.9-7 | 1.7-6</td><td>**−8.5-4** | **−8.0-4** | **−7.2-4** | **−1.8-3**</td><td>14:52 | 8:38 | 9:24 | 2:27:04</td></tr>
<tr><td>tai25b | 973 | 625;</td><td>296,344;18325 | 25000 | 25000 | 25000</td><td>9.9-7 | 2.9-5 | 3.7-5 | 4.2-5</td><td>**−2.7-4** | **−2.0-3** | **−2.4-3** | **−2.5-3**</td><td>1:18:04 | 1:28:33 | 1:55:04 | 2:21:35</td></tr>
<tr><td>tai30a | 1393 | 900;</td><td>39,39;1614 | 25000 | 25000 | 25000</td><td>9.9-7 | 4.7-6 | 4.6-6 | 1.3-5</td><td>−2.3-5 | −6.3-5 | −7.3-5 | **−1.3-4**</td><td>29:11 | 3:53:48 | 6:09:25 | 6:00:13</td></tr>
<tr><td>tai30b | 1393 | 900;</td><td>236,342;16584 | 25000 | 25000 | 25000</td><td>9.9-7 | 2.0-5 | 2.4-5 | 2.6-5</td><td>**−1.8-4** | **−1.0-3** | **−1.2-3** | **−1.2-3**</td><td>2:52:00 | 3:42:12 | 4:28:02 | 5:38:24</td></tr>
<tr><td>tai35a | 1888 | 1225;</td><td>38,38;3467 | 25000 | 25000 | 25000</td><td>9.9-7 | 3.9-6 | 4.0-6 | 1.3-5</td><td>−1.8-5 | −4.8-5 | −5.6-5 | **−1.0-4**</td><td>1:56:18 | 9:21:21 | 15:00:46 | 12:53:01</td></tr>
<tr><td>tai35b | 1888 | 1225;</td><td>142,214;10915 | 25000 | 25000 | 25000</td><td>9.9-7 | 2.1-5 | 2.4-5 | 2.8-5</td><td>**−1.2-4** | **−9.1-4** | **−1.0-3** | **−1.1-3**</td><td>8:01:01 | 8:51:20 | 11:15:52 | 12:51:27</td></tr>
</table>

Table 4 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c|d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>tai40a | 2458 | 1600;</td><td>33,33;3395 | 25000 | 25000 | 25000</td><td>9.9-7 | 3.7-6 | 4.0-6 | 1.4-5</td><td>−1.8-5 | −4.6-5 | −5.3-5 | −1.0-4</td><td>3:56:34 | 20:22:53 | 31:45:29 | 26:00:47</td></tr>
<tr><td>tai40b | 2458 | 1600;</td><td>101,146;7124 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.9-5 | 2.5-5 | 3.1-5</td><td>−1.1-4 | −7.2-4 | −8.1-4 | −8.5-4</td><td>10:55:44 | 17:50:19 | 23:17:25 | 25:23:31</td></tr>
<tr><td>tho30 | 1393 | 900;</td><td>44,74;2925 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.1-5 | 1.5-5 | 2.2-5</td><td>−4.8-5 | −2.6-4 | −3.4-4 | −4.0-4</td><td>1:03:01 | 3:46:49 | 4:46:03 | 5:44:33</td></tr>
<tr><td>tho40 | 2458 | 1600;</td><td>24,51;3998 | 25000 | 25000 | 25000</td><td>9.9-7 | 9.3-6 | 1.3-5 | 2.0-5</td><td>−4.2-5 | −2.1-4 | −2.7-4 | −3.2-4</td><td>5:08:15 | 17:12:50 | 24:35:42 | 26:05:11</td></tr>
<tr><td>be250.1 | 251 | 251;</td><td>122,123;2800 | 4327 | 5345 | 3537</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−4.7-7 | −2.0-7 | −3.6-7 | −4.1-7</td><td>1:13 | 1:16 | 1:35 | 1:37</td></tr>
<tr><td>be250.2 | 251 | 251;</td><td>121,121;2842 | 3827 | 5108 | 3044</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−7.9-7 | −8.6-7 | −5.3-7 | −8.0-7</td><td>1:12 | 1:08 | 1:28 | 1:22</td></tr>
<tr><td>be250.3 | 251 | 251;</td><td>84,89;2200 | 3796 | 4331 | 2592</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−7.7-7 | −1.1-6 | −7.3-7 | −1.1-6</td><td>59 | 1:11 | 1:18 | 1:11</td></tr>
<tr><td>be250.4 | 251 | 251;</td><td>208,209;3850 | 8023 | 8350 | 6453</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.2-6 | −1.1-6 | −1.1-6 | −2.9-7</td><td>1:42 | 2:23 | 2:24 | 2:53</td></tr>
<tr><td>be250.5 | 251 | 251;</td><td>115,127;2791 | 4460 | 5089 | 3174</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−6.5-7 | −7.5-7 | −6.4-7 | −7.2-7</td><td>1:15 | 1:23 | 1:31 | 1:26</td></tr>
<tr><td>bqp500-1 | 501 | 501;</td><td>138,171;2499 | 6473 | 6932 | 4086</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>2.0-6 | −1.4-6 | −3.4-7 | −1.6-6</td><td>5:20 | 8:35 | 9:45 | 9:13</td></tr>
<tr><td>bqp500-2 | 501 | 501;</td><td>142,194;2390 | 8008 | 10582 | 4862</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>4.1-7 | −4.2-7 | −8.6-8 | −1.2-6</td><td>5:29 | 10:46 | 14:42 | 10:52</td></tr>
<tr><td>bqp500-3 | 501 | 501;</td><td>135,180;2390 | 8192 | 8915 | 4965</td><td>9.7-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>7.6-7 | −1.5-6 | 3.7-7 | −5.8-7</td><td>6:31 | 12:53 | 12:25 | 11:22</td></tr>
<tr><td>bqp500-4 | 501 | 501;</td><td>128,174;2390 | 7188 | 9012 | 4031</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>6.1-7 | −1.0-6 | −3.8-7 | −1.2-6</td><td>6:08 | 10:37 | 12:10 | 9:11</td></tr>
<tr><td>bqp500-5 | 501 | 501;</td><td>169,206;2910 | 6898 | 7641 | 4541</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.1-6 | −8.9-7 | −1.2-6 | −8.2-7</td><td>7:25 | 10:34 | 10:57 | 10:19</td></tr>
<tr><td>gka1f | 501 | 501;</td><td>166,203;2780 | 6717 | 8147 | 4600</td><td>9.8-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>5.9-7 | −1.3-6 | −1.2-6 | −5.6-7</td><td>6:32 | 9:38 | 11:28 | 10:31</td></tr>
<tr><td>gka2f | 501 | 501;</td><td>205,242;3541 | 7519 | 8949 | 5403</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.5-6 | −1.5-6 | −1.4-6 | −1.0-6</td><td>7:54 | 10:50 | 12:52 | 12:10</td></tr>
<tr><td>gka3f | 501 | 501;</td><td>174,216;2954 | 6102 | 7037 | 3957</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>6.8-7 | −1.1-6 | −2.0-6 | −1.6-7</td><td>6:51 | 9:07 | 10:46 | 9:07</td></tr>
<tr><td>gka4f | 501 | 501;</td><td>183,222;3101 | 6673 | 7529 | 4070</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>8.2-8 | −1.1-6 | −4.2-7 | −3.5-7</td><td>7:10 | 9:13 | 11:20 | 9:14</td></tr>
<tr><td>gka5f | 501 | 501;</td><td>142,187;2520 | 6482 | 7023 | 4210</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.5-8 | −5.9-7 | −9.4-7 | −7.5-7</td><td>5:53 | 9:14 | 10:36 | 9:45</td></tr>
<tr><td>soybean-large.2 | 308 | 307;</td><td>2, 2;1171 | 1190 | 5050 | 2261</td><td>9.9-7 | 9.2-7 | 9.9-7 | 9.9-7</td><td>−1.0-7 | −7.7-8 | −1.2-7 | −7.0-8</td><td>29 | 29 | 3:45 | 3:09</td></tr>
<tr><td>soybean-large.3 | 308 | 307;</td><td>2, 2;934 | 922 | 5993 | 2159</td><td>7.2-7 | 8.8-7 | 9.6-7 | 8.5-7</td><td>−2.8-7 | −2.1-7 | −5.4-9 | 1.4-7</td><td>25 | 24 | 4:47 | 3:54</td></tr>
<tr><td>soybean-large.4 | 308 | 307;</td><td>52,52;1506 | 1609 | 13512 | 3831</td><td>8.7-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.4-7 | −2.8-7 | −1.2-7 | −1.6-7</td><td>52 | 42 | 10:51 | 7:18</td></tr>
<tr><td>soybean-large.5 | 308 | 307;</td><td>2, 2;814 | 850 | 2974 | 1404</td><td>9.8-7 | 9.7-7 | 9.9-7 | 9.9-7</td><td>−8.4-8 | −9.1-8 | −7.9-8 | −1.7-7</td><td>22 | 23 | 2:23 | 2:05</td></tr>
</table>

Table 4 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c|d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>soybean-large.6 | 308 | 307;</td><td>0, 0;413 | 413 | 545 | 681</td><td>9.4-7 | 9.4-7 | 6.8-7 | 9.1-7</td><td>−1.9-7 | −1.9-7 | 3.5-8 | 1.3-6</td><td>12 | 12 | 21 | 44</td></tr>
<tr><td>spambase-large.2 | 1501 | 1500;</td><td>0, 0;535 | 535 | 992 | 4429</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.3-5 | −1.3-5 | −1.2-5 | −1.3-5</td><td>11:07 | 11:17 | 22:17 | 3:12:36</td></tr>
<tr><td>spambase-large.3 | 1501 | 1500;</td><td>8, 8;1844 | 1705 | 1830 | 6617</td><td>9.9-7 | 9.8-7 | 9.9-7 | 9.9-7</td><td>−7.6-6 | −7.6-6 | −6.6-6 | −3.3-6</td><td>1:40:31 | 35:47 | 58:10 | 6:13:50</td></tr>
<tr><td>spambase-large.4 | 1501 | 1500;</td><td>8, 8;4519 | 3761 | 7091 | 25000</td><td>9.9-7 | 9.9-7 | 9.9-7 | 2.2-2</td><td>−2.6-6 | 9.4-8 | −5.2-7 | −10.0-1</td><td>2:49:39 | 1:19:26 | 5:32:29 | 17:57:38</td></tr>
<tr><td>spambase-large.5 | 1501 | 1500;</td><td>8, 8;9184 | 8398 | 7510 | 25000</td><td>9.7-7 | 9.9-7 | 9.8-7 | 1.1-2</td><td>−3.0-5 | −2.9-5 | −2.4-5 | 3.0-1</td><td>4:49:37 | 3:26:14 | 3:21:11 | 18:14:24</td></tr>
<tr><td>spambase-large.6 | 1501 | 1500;</td><td>8, 8;2798 | 2031 | 2415 | 25000</td><td>9.9-7 | 9.9-7 | 9.9-7 | 1.8-2</td><td>4.9-5 | −4.2-5 | −5.8-5 | −10.0-1</td><td>2:07:59 | 49:32 | 1:07:56 | 17:07:48</td></tr>
<tr><td>abalone-large.2 | 1001 | 1000;</td><td>0, 0;576 | 576 | 650 | 1493</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>1.2-5 | 1.2-5 | 6.6-6 | −1.4-6</td><td>5:01 | 5:07 | 5:08 | 31:13</td></tr>
<tr><td>abalone-large.3 | 1001 | 1000;</td><td>21,21;762 | 765 | 796 | 1306</td><td>9.2-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−2.1-6 | −3.6-6 | −9.9-7 | −4.2-6</td><td>7:29 | 6:09 | 8:56 | 22:21</td></tr>
<tr><td>abalone-large.4 | 1001 | 1000;</td><td>0, 0;545 | 545 | 629 | 710</td><td>9.9-7 | 9.9-7 | 9.6-7 | 9.9-7</td><td>1.9-6 | 1.9-6 | −6.9 − 6 | −9.2-7</td><td>6:43 | 6:50 | 5:01 | 12:03</td></tr>
<tr><td>abalone-large.5 | 1001 | 1000;</td><td>38,38;797 | 834 | 1107 | 833</td><td>9.5-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−2.2-5 | −1.5-5 | −2.1-5 | −2.1-5</td><td>11:45 | 8:39 | 9:11 | 14:17</td></tr>
<tr><td>abalone-large.6 | 1001 | 1000;</td><td>8, 8;781 | 796 | 1101 | 950</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.4-5 | −1.4-5 | −1.8-5 | −1.9-5</td><td>9:12 | 8:21 | 8:49 | 15:24</td></tr>
<tr><td>segment-large.2 | 1001 | 1000;</td><td>8, 8;1191 | 1264 | 1080 | 1745</td><td>9.9-7 | 9.9-7 | 9.8-7 | 9.9-7</td><td>4.6-6 | 5.0-6 | −4.7 − 6 | −5.0-7</td><td>9:16 | 9:15 | 8:27 | 34:22</td></tr>
<tr><td>segment-large.3 | 1001 | 1000;</td><td>0, 0;373 | 373 | 412 | 1956</td><td>9.9-7 | 9.9-7 | 9.8-7 | 9.9-7</td><td>1.8-6 | 1.8-6 | −7.1-7 | −1.1-6</td><td>2:43 | 2:41 | 3:33 | 37:08</td></tr>
<tr><td>segment-large.4 | 1001 | 1000;</td><td>2, 2;1879 | 2024 | 19479 | 6354</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−5.8-7 | −5.5-7 | −4.5-7 | −5.0-7</td><td>13:52 | 14:50 | 5:23:13 | 3:07:06</td></tr>
<tr><td>segment-large.5 | 1001 | 1000;</td><td>8, 8;2449 | 2711 | 22003 | 8257</td><td>9.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−6.2-7 | −6.7-7 | −6.0-7 | −6.4-7</td><td>19:06 | 20:31 | 6:09:59 | 4:19:44</td></tr>
<tr><td>segment-large.6 | 1001 | 1000;</td><td>8, 8;3158 | 3262 | 25000 | 10211</td><td>9.9-7 | 9.9-7 | 1.3-6 | 9.9-7</td><td>−1.5-6 | −1.5-6 | −9.6-7 | −1.0-6</td><td>24:00 | 24:06 | 7:10:04 | 5:25:59</td></tr>
<tr><td>housing.2 | 507 | 506;</td><td>8, 8;3373 | 3284 | 2679 | 2566</td><td>9.9-7 | 9.6-7 | 9.9-7 | 8.6-7</td><td>−5.9-6 | −5.4-6 | −5.2-6 | −5.3-6</td><td>4:50 | 4:31 | 3:26 | 7:52</td></tr>
<tr><td>housing.3 | 507 | 506;</td><td>8, 8;1576 | 1247 | 1523 | 1338</td><td>9.7-7 | 9.9-7 | 9.9-7 | 9.8-7</td><td>1.7-6 | 8.0-6 | −6.7-6 | 5.2-6</td><td>3:20 | 1:34 | 1:56 | 4:29</td></tr>
<tr><td>housing.4 | 507 | 506;</td><td>8, 8;1645 | 1368 | 1064 | 1090</td><td>9.9-7 | 9.9-7 | 9.9-7 | 8.4-7</td><td>−4.0-6 | −3.5-6 | −4.9-6 | 8.3-5</td><td>2:50 | 2:00 | 1:25 | 3:40</td></tr>
<tr><td>housing.5 | 507 | 506;</td><td>8, 8;1918 | 1319 | 1916 | 1451</td><td>9.9-7 | 9.6-7 | 9.3-7 | 8.8-7</td><td>3.3-5 | −3.2-5 | 3.6-5 | 6.3-5</td><td>3:30 | 2:07 | 2:36 | 5:03</td></tr>
<tr><td>housing.6 | 507 | 506;</td><td>11,11;533 | 536 | 842 | 1958</td><td>9.9-7 | 9.9-7 | 9.8-7 | 9.5-7</td><td>−1.2-6 | −9.7-6 | 5.9-6 | 6.3-5</td><td>1:06 | 53 | 1:20 | 6:29</td></tr>
</table>

Table 5 Performance of SDPNAL+ (a), ADMM+ (b), SDPAD (c) and 2EBD (d) on θ and R1TA problems ($\varepsilon = 10^{-6}$)

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c| d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>theta10 | 12470 | 500;</td><td>11,11;200 | 396 | 333 | 422</td><td>7.6-7 | 9.1-7 | 9.9-7 | 9.8-7</td><td>6.7-8 | −2.1-6 | −1.4-6 | 1.2-6</td><td>32 | 51 | 36 | 59</td></tr>
<tr><td>theta102 | 37467 | 500;</td><td>11,11;84 | 159 | 127 | 312</td><td>6.8-7 | 9.1-7 | 9.2-7 | 9.9-7</td><td>−9.3-8 | 1.3-6 | 1.7-6 | 1.6-6</td><td>21 | 22 | 21 | 47</td></tr>
<tr><td>theta103 | 62516 | 500;</td><td>20,20;64 | 144 | 104 | 300</td><td>8.1-7 | 9.7-7 | 9.9-7 | 9.8-7</td><td>−4.1-8 | −6.2-8 | 2.0-7 | 2.0-6</td><td>38 | 20 | 21 | 45</td></tr>
<tr><td>theta104 | 87245 | 500;</td><td>43,47;63 | 151 | 116 | 342</td><td>4.4-7 | 9.0-7 | 9.5-7 | 9.8-7</td><td>−5.5-7 | −2.4-8 | 8.3-7 | 3.2-6</td><td>53 | 22 | 19 | 51</td></tr>
<tr><td>theta12 | 17979 | 600;</td><td>13,13;200 | 413 | 358 | 444</td><td>3.7-7 | 8.4-7 | 8.5-7 | 9.9-7</td><td>5.6-8 | 2.1-6 | 1.0-6 | 1.1-6</td><td>51 | 1:24 | 1:02 | 1:38</td></tr>
<tr><td>theta123 | 90020 | 600;</td><td>12,12;70 | 157 | 105 | 325</td><td>7.1-7 | 9.2-7 | 9.2-7 | 9.7-7</td><td>1.9-8 | −6.8-8 | 2.2-7 | 2.5-6</td><td>36 | 35 | 34 | 1:19</td></tr>
<tr><td>hamming-9-8 | 2305 | 512;</td><td>12,12;200 | 2635 | 3129 | 1276</td><td>2.5-7 | 9.8-7 | 9.7-7 | 9.4-7</td><td>−9.2-8 | −1.2-5 | 5.6-7 | −5.8-6</td><td>21 | 3:17 | 4:07 | 2:02</td></tr>
<tr><td>hamming-10-2 | 23041 | 1024;</td><td>10,10;200 | 667 | 731 | 1066</td><td>4.8-7 | 6.9-7 | 9.6-7 | 9.9-7</td><td>−5.9-7 | −1.5-5 | 2.2-6 | 2.5-5</td><td>2:54 | 7:19 | 9:39 | 12:05</td></tr>
<tr><td>hamming-9-5-6 | 53761 | 512;</td><td>6, 6;200 | 1022 | 1215 | 197</td><td>6.2-7 | 8.8-7 | 9.8-7 | 8.8-7</td><td>−1.7-7 | −1.1-5 | −1.8-6 | −5.1-6</td><td>20 | 1:24 | 1:36 | 20</td></tr>
<tr><td>G43 | 9991 | 1000;</td><td>27,27;200 | 1237 | 1097 | 962</td><td>9.8-7 | 9.4-7 | 9.8-7 | 9.9-7</td><td>8.9-7 | −3.5-6 | −1.8-6 | 2.0-6</td><td>3:32 | 9:54 | 9:13 | 12:49</td></tr>
<tr><td>G44 | 9991 | 1000;</td><td>30,30;200 | 1236 | 1110 | 996</td><td>6.1-7 | 9.7-7 | 9.3-7 | 9.9-7</td><td>5.9-9 | −3.6-6 | −8.8-7 | 1.6-6</td><td>3:48 | 9:57 | 9:15 | 13:17</td></tr>
<tr><td>G45 | 9991 | 1000;</td><td>26,26;200 | 1261 | 1120 | 1007</td><td>6.4-7 | 9.9-7 | 9.6-7 | 9.9-7</td><td>3.8-8 | 3.2-6 | 1.8-6 | 1.6-6</td><td>3:31 | 10:04 | 9:21 | 13:35</td></tr>
<tr><td>G46 | 9991 | 1000;</td><td>28,30;200 | 1284 | 1142 | 974</td><td>7.9-7 | 9.6-7 | 9.9-7 | 9.9-7</td><td>6.2-8 | −3.1-6 | −1.6-6 | 1.3-6</td><td>3:49 | 10:11 | 9:21 | 12:55</td></tr>
<tr><td>G47 | 9991 | 1000;</td><td>30,31;200 | 1267 | 1088 | 1030</td><td>3.7-7 | 9.3-7 | 9.5-7 | 9.9-7</td><td>1.5-7 | 2.8-6 | 8.9-7 | 1.2-6</td><td>3:52 | 9:59 | 8:51 | 13:47</td></tr>
<tr><td>G51 | 5910 | 1000;</td><td>148,584;200 | 6151 | 10210 | 8746</td><td>7.3-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−7.9-8 | −1.7-7 | 8.6-8 | 1.9-7</td><td>41:06 | 53:54 | 1:33:48 | 1:55:48</td></tr>
<tr><td>G52 | 5917 | 1000;</td><td>458,1619;200 | 25000 | 25000 | 25000</td><td>9.3-7 | 1.6-6 | 3.5-6 | 2.9-6</td><td>2.0-6 | 7.1-6 | 1.4-5 | 1.5-5</td><td>3:53:19 | 3:24:55 | 3:45:22 | 5:38:42</td></tr>
<tr><td>G53 | 5915 | 1000;</td><td>425,1183;200 | 25000 | 25000 | 25000</td><td>9.9-7 | 1.5-6 | 3.7-6 | 3.7-6</td><td>1.4-6 | 5.9-6 | 1.5-5 | 1.8-5</td><td>2:16:35 | 3:12:47 | 3:24:18 | 5:30:23</td></tr>
<tr><td>G54 | 5917 | 1000;</td><td>123,462;200 | 3892 | 5633 | 5398</td><td>9.3-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>4.2-8 | −2.8-6 | 3.3-7 | 3.2-7</td><td>23:17 | 33:11 | 49:11 | 1:13:51</td></tr>
<tr><td>1dc.1024 | 24064 | 1024;</td><td>48,74;200 | 5077 | 9728 | 15069</td><td>9.1-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>3.4-6 | 4.2-6 | 4.9-6 | 4.0-6</td><td>14:32 | 50:49 | 1:31:42 | 3:03:20</td></tr>
<tr><td>1et.1024 | 9601 | 1024;</td><td>64,129;200 | 3956 | 10174 | 17252</td><td>5.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>2.2-6 | 3.1-6 | 3.0-6 | 2.7-6</td><td>35:33 | 50:14 | 1:45:03 | 3:53:32</td></tr>
<tr><td>1tc.1024 | 7937 | 1024;</td><td>156,417;200 | 5775 | 25000 | 18474</td><td>7.5-7 | 9.9-7 | 2.3-6 | 9.9-7</td><td>8.9-7 | 3.8-6 | 3.3-6 | 2.3-6</td><td>1:22:24 | 54:35 | 4:13:43 | 3:47:44</td></tr>
<tr><td>1zc.1024 | 16641 | 1024;</td><td>16,16;200 | 884 | 734 | 4488</td><td>8.4-7 | 9.7-7 | 9.2-7 | 9.6-7</td><td>2.7-8 | 6.9-7 | 1.6-6 | 2.4-5</td><td>4:56 | 12:44 | 8:49 | 49:10</td></tr>
<tr><td>2dc.1024 | 169163 | 1024;</td><td>148,376;200 | 6951 | 14316 | 23007</td><td>9.2-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>7.8-6 | 3.1-5 | 2.6-5 | 2.6-5</td><td>2:21:20 | 1:24:20 | 4:20:36 | 5:32:26</td></tr>
<tr><td>1dc.2048 | 58368 | 2048;</td><td>62,112;200 | 5520 | 12938 | 20527</td><td>9.7-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>5.5-6 | 6.6-6 | 6.7-6 | 6.8-6</td><td>1:55:31 | 6:10:05 | 14:00:08 | 28:29:05</td></tr>
</table>

Table 5 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c| d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>1et.2048 | 22529 | 2048;</td><td>228,658;200 | 3601 | 13985 | 25000</td><td>8.7-7 | 9.9-7 | 9.9-7 | **6.1-3**</td><td>3.9-6 | 4.3-6 | 4.2-6 | **2.2-2**</td><td>5:29:46 | 3:59:47 | 17:25:13 | 40:08:45</td></tr>
<tr><td>1tc.2048 | 18945 | 2048;</td><td>509,1725;200 | 6574 | 20819 | 25000</td><td>8.2-7 | 9.9-7 | 9.9-7 | 1.2-6</td><td>9.9-7 | 4.8-6 | 3.8-6 | 5.5-6</td><td>22:10:45 | 7:26:56 | 23:37:33 | 39:35:04</td></tr>
<tr><td>2dc.2048 | 504452 | 2048;</td><td>167,385;200 | 6293 | 25000 | 16945</td><td>9.8-7 | 9.9-7 | 3.7-6 | 9.9-7</td><td>1.8-5 | 2.0-5 | 3.5-5 | 2.8-5</td><td>14:22:06 | 8:52:47 | 47:15:41 | 28:09:34</td></tr>
<tr><td>nonsym(8,4) | 46655 | 512;</td><td>13,13;200 | 6927 | 6871 | 25000</td><td>1.5-7 | 8.7-7 | 9.9-7 | **1.9-2**</td><td>−4.1-7 | 5.7-6 | 2.1-5 | **6.2-1**</td><td>29 | 8:40 | 8:54 | 1:10:07</td></tr>
<tr><td>nonsym(9,4) | 91124 | 729;</td><td>25,33;200 | 25000 | 4584 | 25000</td><td>4.2-7 | 3.2-5 | 9.7-7 | **1.9-2**</td><td>7.4-6 | **5.2-4** | −1.7-5 | **4.2-3**</td><td>2:00 | 1:18:21 | 14:29 | 2:41:00</td></tr>
<tr><td>nonsym(10,4) | 166374 | 1000;</td><td>17,21;200 | 25000 | 6711 | 25000</td><td>7.9-7 | 2.8-5 | 9.9-7 | **1.6-2**</td><td>−2.5-6 | **−1.9-4** | 2.6-5 | **6.2-1**</td><td>3:37 | 2:57:30 | 48:32 | 5:52:44</td></tr>
<tr><td>nonsym(11,4) | 287495 | 1331;</td><td>19,22;200 | 25000 | 16627 | 25000</td><td>2.5-7 | **1.3-3** | 9.9-7 | **9.9-3**</td><td>7.5-6 | **5.1-2** | 3.0-5 | **−2.2-1**</td><td>7:14 | 6:30:10 | 4:57:08 | 12:51:58</td></tr>
<tr><td>nonsym(5,5) | 50624 | 625;</td><td>30,31;200 | 12638 | 4918 | 25000</td><td>2.4-7 | 9.6-7 | 9.9-7 | **2.5-3**</td><td>3.7-6 | 9.0-6 | 1.8-5 | **−4.8-2**</td><td>1:18 | 27:19 | 10:10 | 1:50:10</td></tr>
<tr><td>nonsym(6,5) | 194480 | 1296;</td><td>26,28;200 | 25000 | 11981 | 25000</td><td>6.6-7 | **1.6-4** | 9.9-7 | **1.6-3**</td><td>2.0-5 | **−3.0-3** | −2.8-5 | **−4.9-2**</td><td>6:39 | 5:57:28 | 2:59:30 | 11:24:24</td></tr>
<tr><td>sym_rd(3,35) | 82250 | 666;</td><td>43,46;72 | 2964 | 2812 | 11937</td><td>5.4-7 | 9.7-7 | 9.9-7 | 9.5-7</td><td>−8.0-6 | −1.3-6 | 1.8-5 | −1.9-6</td><td>1:27 | 7:54 | 7:07 | 1:09:55</td></tr>
<tr><td>sym_rd(3,40) | 135750 | 861;</td><td>37,39;82 | 3736 | 3356 | 25000</td><td>5.7-7 | 9.4-7 | 9.9-7 | **3.9-3**</td><td>−1.6-5 | −1.2-6 | −2.1-5 | **1.3-1**</td><td>2:18 | 19:34 | 16:55 | 4:37:17</td></tr>
<tr><td>sym_rd(3,45) | 211875 | 1081;</td><td>31,31;87 | 4689 | 4498 | 25000</td><td>5.5-7 | 9.3-7 | 9.9-7 | **5.5-3**</td><td>5.7-6 | −3.5-6 | −3.9-5 | **−1.5-1**</td><td>4:31 | 46:05 | 41:11 | 8:09:46</td></tr>
<tr><td>sym_rd(3,50) | 316250 | 1326;</td><td>37,37;87 | 4432 | 4161 | 25000</td><td>7.6-7 | 9.1-7 | 9.6-7 | **3.2-3**</td><td>2.4-5 | −3.7-6 | 4.2-5 | **1.1-1**</td><td>8:24 | 1:13:12 | 1:07:37 | 13:45:47</td></tr>
<tr><td>sym_rd(4,35) | 73814 | 630;</td><td>46,51;77 | 969 | 3400 | 25000</td><td>2.2-7 | 9.9-7 | 9.7-7 | **5.1-4**</td><td>6.2-6 | −4.2-6 | −4.2-5 | **−1.7-2**</td><td>1:45 | 4:44 | 9:10 | 1:58:12</td></tr>
<tr><td>sym_rd(4,40) | 123409 | 820;</td><td>60,76;86 | 447 | 761 | 2396</td><td>6.6-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−7.7-6 | −1.3-5 | −1.3-5 | −1.3-5</td><td>4:56 | 3:37 | 7:06 | 21:58</td></tr>
<tr><td>sym_rd(4,45) | 194579 | 1035;</td><td>45,62;90 | 462 | 737 | 2569</td><td>3.5-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−4.7-6 | −1.4-5 | −1.4-5 | −1.4-5</td><td>7:44 | 6:56 | 12:42 | 43:48</td></tr>
<tr><td>sym_rd(4,50) | 292824 | 1275;</td><td>45,62;91 | 466 | 758 | 2824</td><td>7.9-7 | 9.9-7 | 9.9-7 | 9.9-7</td><td>−1.3-5 | −1.6-5 | −1.6-5 | −1.6-5</td><td>12:44 | 12:11 | 22:43 | 1:21:38</td></tr>
<tr><td>sym_rd(5,15) | 54263 | 816;</td><td>41,43;111 | 1549 | 1980 | 25000</td><td>2.4-7 | 8.8-7 | 9.7-7 | **3.2-4**</td><td>8.2-6 | 7.1-5 | −3.7-5 | **1.6-2**</td><td>2:53 | 6:46 | 7:53 | 3:28:56</td></tr>
<tr><td>sym_rd(5,20) | 230229 | 1771;</td><td>29,35;139 | 2832 | 3563 | 25000</td><td>3.1-7 | 9.3-7 | 9.8-7 | **5.5-3**</td><td>−1.5-5 | 9.7-5 | **−1.0-4** | **−3.1-1**</td><td>27:28 | 1:56:14 | 2:22:52 | 27:04:51</td></tr>
<tr><td>sym_rd(6,15) | 38759 | 680;</td><td>32,35;137 | 1358 | 1652 | 13111</td><td>6.4-7 | 9.3-7 | 9.9-7 | 9.9-7</td><td>2.6-5 | 5.2-5 | −6.9-5 | −6.3-7</td><td>1:33 | 3:52 | 4:26 | 1:12:00</td></tr>
<tr><td>sym_rd(6,20) | 177099 | 1540;</td><td>26,37;185 | 3280 | 3576 | 25000</td><td>6.4-7 | 9.7-7 | 9.6-7 | **3.0-4**</td><td>3.0-6 | **1.5-4** | **−1.1-4** | **−2.5-2**</td><td>22:48 | 1:34:25 | 1:43:31 | 17:41:06</td></tr>
<tr><td>nsym_rd([20,25,25]) | 68249 | 500;</td><td>47,51;66 | 5883 | 6641 | 25000</td><td>6.0-7 | 9.8-7 | 9.7-7 | **6.1-3**</td><td>1.0-5 | −3.0-5 | 2.5-5 | **−1.2-1**</td><td>58 | 7:55 | 8:10 | 1:12:25</td></tr>
<tr><td>nsym_rd([25,20,25]) | 68249 | 500;</td><td>49,50;77 | 6113 | 6912 | 25000</td><td>7.0-7 | 9.9-7 | 9.9-7 | **5.5-3**</td><td>−1.0-5 | −3.0-5 | −1.3-5 | **−8.7-2**</td><td>58 | 7:26 | 8:25 | 1:11:35</td></tr>
<tr><td>nsym_rd([25,25,20]) | 68249 | 500;</td><td>14,14;129 | 6252 | 6898 | 25000</td><td>9.0-8 | 9.9-7 | 9.9-7 | **2.8-3**</td><td>−1.6-6 | −5.9-6 | 2.7-5 | **5.2-2**</td><td>34 | 8:11 | 8:31 | 1:10:17</td></tr>
</table>

Table 5 continued

<table>
<tr><th>Problem | m | n_s; n_l</th><th>Iteration
a| b| c| d</th><th>η
a| b| c| d</th><th>η_g
a| b| c| d</th><th>Time
a| b| c| d</th></tr>
<tr><td>nsym_rd([25,25,25]) | 105624 | 625;</td><td>56,64;95 | 11250 | 3705 | 25000</td><td>9.0-7 | 9.9-7 | 9.9-7 | 6.6-4</td><td>−1.7-5 | −4.9-6 | −1.3-5 | 1.3-2</td><td>1:33 | 26:20 | 8:09 | 2:04:05</td></tr>
<tr><td>nsym_rd([30,30,30]) | 216224 | 900;</td><td>30,33;122 | 25000 | 4829 | 25000</td><td>6.2-7 | 2.0-5 | 9.9-7 | 4.9-3</td><td>2.7-6 | 5.5-4 | 3.2-5 | 1.5-1</td><td>3:52 | 2:21:43 | 26:48 | 5:07:41</td></tr>
<tr><td>nsym_rd([35,35,35]) | 396899 | 1225;</td><td>45,49;141 | 25000 | 8788 | 25000</td><td>2.0-7 | 1.4-3 | 9.9-7 | 1.0-3</td><td>−3.5-6 | 4.1-3 | −4.2-5 | 3.6-3</td><td>9:14 | 5:18:55 | 1:57:42 | 11:13:11</td></tr>
<tr><td>nsym_rd([40,40,40]) | 672399 | 1600;</td><td>33,35;93 | 25000 | 25000 | 25000</td><td>3.3-7 | 1.1-4 | 3.7-4 | 5.1-4</td><td>1.0-5 | 5.0-3 | 1.3-2 | −1.9-2</td><td>14:23 | 12:12:03 | 13:56:21 | 22:41:13</td></tr>
<tr><td>nsym_rd([8,8,8,8]) | 46655 | 512;</td><td>11,11;200 | 5325 | 5865 | 25000</td><td>3.3-7 | 9.4-7 | 9.9-7 | 1.2-3</td><td>−6.6-6 | 6.9-6 | −1.3-5 | 1.3-2</td><td>28 | 7:09 | 7:13 | 1:12:01</td></tr>
<tr><td>nsym_rd([9,9,9,9]) | 91124 | 729;</td><td>14,14;200 | 21833 | 4073 | 25000</td><td>3.1-7 | 9.0-7 | 9.6-7 | 1.6-2</td><td>6.9-6 | −7.1-6 | −3.1-5 | −2.5-1</td><td>1:07 | 1:12:18 | 12:47 | 2:52:40</td></tr>
<tr><td>nonsym(12,4) | 474551 | 1728;</td><td>5,17;200 | 16473 | 25000 | 25000</td><td>2.8-8 | 8.8-7 | 1.2-2 | 1.2-2</td><td>−1.1-7 | −2.7-6 | −7.7-2 | 5.8-1</td><td>16:55 | 9:04:03 | 15:26:14 | 24:24:33</td></tr>
<tr><td>nonsym(13,4) | 753570 | 2197;</td><td>15,55;200 | 25000 | 25000 | 25000</td><td>5.2-7 | 5.4-4 | 9.1-3 | 1.9-2</td><td>−1.8-7 | 3.6-2 | −1.6-1 | 2.7-1</td><td>1:51:34 | 29:11:11 | 32:02:36 | 54:30:04</td></tr>
<tr><td>nonsym(7,5) | 614655 | 2401;</td><td>32,43;200 | 25000 | 25000 | 25000</td><td>2.4-7 | 1.4-3 | 1.2-2 | 1.8-2</td><td>8.7-6 | −5.7-2 | −1.2-1 | −1.1-1</td><td>53:29 | 38:36:39 | 43:46:36 | 67:40:52</td></tr>
<tr><td>nonsym(8,5) | 1679615 | 4096;</td><td>14,22;200 | 12791 | 10732 | 7851</td><td>5.2-7 | 1.2-3 | 1.3-2 | 1.2-2</td><td>−5.5-6 | −3.7-2 | 2.5-1 | −4.3-1</td><td>2:46:20 | 99:00:46 | 99:01:08 | 99:03:37</td></tr>
<tr><td>nonsym(18,4) | 5000210 | 5832;</td><td>13,55;200 | 8748 | 7962 | 7017</td><td>5.8-7 | 3.5-4 | 7.8-3 | 1.4-2</td><td>2.6-5 | −1.3-2 | −3.5-1 | 3.8-1</td><td>8:50:14 | 99:02:10 | 99:01:13 | 99:05:37</td></tr>
<tr><td>nonsym(20,4) | 9260999 | 8000;</td><td>7,17;200 | 3231 | 3031 | 2645</td><td>7.4-8 | 4.7-4 | 9.6-3 | 2.2-2</td><td>−6.4-6 | 5.7-2 | −4.4-1 | −3.5-1</td><td>8:26:40 | 99:07:11 | 99:03:17 | 99:16:28</td></tr>
<tr><td>nonsym(21,4) | 12326390 | 9261;</td><td>7,21;200 | 1918 | 1904 | 1792</td><td>5.7-8 | 2.7-4 | 9.8-3 | 5.2-3</td><td>−1.2-6 | 2.6-3 | 5.0-1 | 9.4-1</td><td>14:22:05 | 99:09:25 | 99:05:25 | 99:29:18</td></tr>
</table>

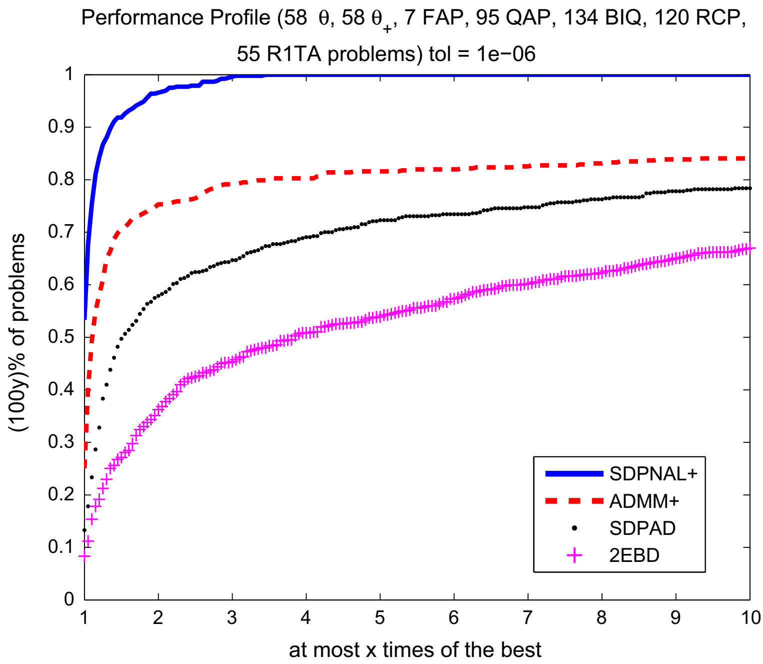

Fig. 1 Performance profiles of SDPNAL+, ADMM+, SDPAD and 2EBD on [1,10]

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References

1. Arnold, V.I.: On matrices depending on parameters. Russ. Math. Surv. **26**, 29–43 (1971)
2. Bonnans, J.F., Shapiro, A.: Perturbation Analysis of Optimization Problems. Springer, Berlin (2000)
3. Borwein, J.M., Lewis, A.S.: Convex Analysis and Nonlinear Optimization: Theory and Examples, vol. 3. Springer, Berlin (2006)
4. Burer, S.: On the copositive representation of binary and continuous nonconvex quadratic programs. Math. Program. **120**, 479–495 (2009)
5. Burer, S., Monteiro, R.D., Zhang, Y.: A computational study of a gradient-based log-barrier algorithm for a class of large-scale sdps. Math. Program. **95**, 359–379 (2003)
6. Chen, C., He, B., Ye, Y., Yuan, X.: The direct extension of admm for multi-block convex minimization problems is not necessarily convergent. Math. Program. (to appear)
7. Eisenblätter, A., Grötschel, M., Koster, A.M.: Frequency planning and ramifications of coloring. Discuss. Math. Graph Theory **22**, 51–88 (2002)
8. Hahn, P., Anjos, M.: QAPLIB—a quadratic assignment problem library. http://www.seas.upenn.edu/qaplib
9. Hiriart-Urruty, J.-B., Strodiot, J.-J., Nguyen, V.H.: Generalized Hessian matrix and second-order optimality conditions for problems with $C^{1,1}$ data. Appl. Math. Optim. **11**, 43–56 (1984)
10. Monteiro, R., Ortiz, C., Svaiter, B.: A first-order block-decomposition method for solving two-easy-block structured semidefinite programs. Math. Program. Comput. **6**, 103–150 (2014)
11. Moreau, J.J.: Décomposition orthogonale d'un espace hilbertien selon deux cones mutuellement polaires. C. R. Acad. Sci. **255**, 238–240 (1962)

12. Nie, J., Wang, L.: Regularization methods for SDP relaxations in large-scale polynomial optimization. SIAM J. Optim. **22**, 408–428 (2012)
13. Nie, J., Wang, L.: Semidefinite relaxations for best rank-1 tensor approximations. SIAM J. Matrix Anal. Appl. **35**, 1155–1179 (2014)
14. Pang, J.-S., Sun, D.-F., Sun, J.: Semismooth homeomorphisms and strong stability of semidefinite and lorentz complementarity problems. Math. Oper. Res. **28**, 39–63 (2003)
15. Peng, J., Wei, Y.: Approximating k-means-type clustering via semidefinite programming. SIAM J. Optim. **18**, 186–205 (2007)
16. Povh, J., Rendl, F.: Copositive and semidefinite relaxations of the quadratic assignment problem. Discret. Optim. **6**, 231–241 (2009)
17. Rockafellar, R.T.: Conjugate duality and optimization. CBMS-NSF Regional Conf. Ser. Appl. Math. vol. 16. SIAM, Philadelphia (1974)
18. Rockafellar, R.T.: Augmented Lagrangians and applications of the proximal point algorithm in convex programming. Math. Oper. Res. **1**, 97–116 (1976)
19. Rockafellar, R.T.: Monotone operators and the proximal point algorithm. SIAM J. Control Optim. **14**, 877–898 (1976)
20. Sloane, N.: Challenge problems: independent sets in graphs. http://www.research.att.com/njas/doc/graphs.html (2005)
21. Sun, D.-F., Sun, J.: Semismooth matrix-valued functions. Math. Oper. Res. **27**, 150–169 (2002)
22. Sun, D.-F., Toh, K.-C., Yang, L.-Q.: A convergent 3-block semi-proximal alternating direction method of multipliers for conic programming with 4-type constraints. SIAM J. Optim. (to appear)
23. Toh, K.C.: Solving large scale semidefinite programs via an iterative solver on the augmented systems. SIAM J. Optim. **14**, 670–698 (2004)
24. Trick, M., Chvatal, V., Cook, B., Johnson, D., McGeoch, C., Tarjan, R.: The second dimacs implementation challenge—NP hard problems: maximum clique, graph coloring, and satisfiability. http://dimacs.rutgers.edu/Challenges/ (1992)
25. Wen, Z., Goldfarb, D., Yin, W.: Alternating direction augmented Lagrangian methods for semidefinite programming. Math. Program. Comput. **2**, 203–230 (2010)
26. Zhao, X.-Y., Sun, D.-F., Toh, K.-C.: A Newton-CG augmented Lagrangian method for semidefinite programming. SIAM J. Optim. **20**, 1737–1765 (2010)